# Exercise set I, Algorithms and Networks 2012 / 2013

This is the first exercise set for the class Algorithms and Networks of the year 2012-2013. Please note:

- Hand in your work on paper. In special circumstances, you can email your work to H.L.Bodlaender@uu.nl. In such a case, have [AN1] in the header.
- Write clear and neat. Messy work can be graded with a 0.
- Write always your name and student number.
- Deadline for this set: monday September 20, 2012.

This exercise set partly reviews some important background for Algorithms and Networks. You may want or need to review some background material of earlier courses, e.g., material on Depth First Search.

We denote the number of vertices of $G$ always with $n$ and the number of edges with $m$.

**Definition** Let $G = (V, A)$ be a graph (directed or undirected). Let $W \subseteq V$ be a set of vertices. The subgraph of $G$, *induced* by $W$ is denoted by $G[W]$, and is formed by taking all vertices in $W$ and all edges/arc between vertices in $W$.

**Definition** A directed graph $G = (V, A)$ is *strongly connected*, if for each pair of vertices $v$, $w$, there is a path in $G$ from $v$ to $w$ and a path from $w$ to $v$.

**Definition** The *strongly connected component* in a graph $G = (V, A)$ of a vertex $v$ is the set of all vertices $w \in V$ such that there is a path from $v$ to $w$ and a path from $w$ to $v$ in $G$.

**Definition** Two vertices $v$ and $w$ are *mutually reachable* in a graph $G$, if there is a path from $v$ to $w$ and a path from $w$ to $v$.

**1. Strongly connected components and strongly connected graphs**

The notion of strongly connected component is an important notion in graph and network theory and graph and network algorithms. Throughout this exercise, $G$ is a *directed* graph.

(i) Suppose that $v$ and $w$ are mutually reachable, and that $w$ and $x$ are mutually reachable. Then show that $v$ and $x$ are mutually reachable.

(ii) Suppose $X$ is the strongly connected component of $v$. Show that for each $w \in X$, $X$ is also the strongly connected component of $w$.

(iii) Explain how you can solve the following problem in $O(n+m)$ time. Given is a graph $G = (V, A)$, and a vertex $v \in V$. Compute the set $\{w \in V \mid$ there is a path from $v$ to $w$ in $G\}$.

(iv) Explain how you can solve the following problem in $O(n+m)$ time. Given is a graph $G = (V, A)$, and a vertex $v \in V$. Compute the set $\{w \in V \mid$ there is a path from $w$ to $v$ in $G\}$.

(v) Explain how you can test in $O(n + m)$ time if a graph is strongly connected.

(vi) Explain how you can compute in $O(n + m)$ time the strongly connected components of $G$.

(vii) What are the strongly connected components of a directed acyclic graph?

**2. Number of paths in directed acyclic graphs** Suppose we have a directed acyclic graph $G = (V, A)$, with two specified vertices $v$, $w \in V$. Give an efficient algorithm to compute the *number of paths* from $v$ to $w$. Analyze the running time of your algorithm. (Your algorithm does not need to give all paths, just the number.)

(Hint: an algorithmic strategy based on dynamic programming gives a fast algorithm here...)

**3. Number of shortest paths in an undirected graph** Exercise 10 from Algorithm Design by Kleinberg and Tardós, page 110. (In the book, an application for social networks is described.)

Suppose we have an undirected graph $G = (V, E)$, and two vertices $s$ and $t$. Compute the number of shortest paths from $s$ to $t$ in $G$.

Remark: in this exercise, the length of a path is just the number of hops, i.e., arcs that are used. Your teacher thinks that this is the hardest exercise of this set.