Exercise set 4, Algorithms and Networks 2012-2013

Please note:

- Hand in your work on paper. In special circumstances, you can email your work to H.L.Bodlaender@uu.nl. In such a case, have [AN4] in the header.
- Write clear and neat. Messy work can be graded with a 0.
- Write always your name and student number.
- Deadline for this set: Monday October 15, 2012.

Bonus questions are not obligatory, and can give you extra points. Answers to research questions are not known to Hans Bodlaender; research questions are also not obligatory and can give you extra points.

1. First question. 1 point Suppose we have an algorithm that solves the (standard) maximum flow problem in $O(n^3)$ time for graphs with n vertices.

If we use this algorithm as subroutine, how much time does it cost to solve the maximum flow problem where we also have lower bounds on capacities. (I.e., look at the method discussed in class. How much time would it take?)

2. Escape problem. 3 point (Exercise 26-1 from *Introduction to Algorithms*, 2nd edition. Cormen, Leiserson, Rivest, Stein.)

An $n \times n$ grid is an undirected graph consisting of n rows and n columns of vertices as shown in the figure. We denote the vertex in the ith row and jth column by $v_{i,j}$. All vertices in a grid have exactly four neighbors, except for the *boundary* points, which are the vertices $v_{i,j}$ with $i = 1$, $i = n$, $j = 1$ or $j = n$.

Given $m \leq n^2$ starting points $v_{x_1,y_1}, \ldots, v_{x_m,y_m}$ in the grid, the *escape problem* is to determine if there are m vertex disjoint paths from the starting points to any m different points in the boundary. For example, part (a) of the figure has an escape, but part (b) does not have an escape.

1. Consider a flow network in which vertices as well as edges have capacities. That is, the total positive flow entering any given vertex is subject to a capacity constraint. Show that determining the maximum flow in a network with edge and vertex constraints can be reduced to an ordinary maximum-flow problem on a flow network of comparable size.
2. Describe an efficient algorithm to solve the escape problem and analyze its running time.

3. Maximum flow by scaling. 4 points Exercise 26-5 from *Introduction to Algorithms*, 2nd edition. Cormen, Leiserson, Rivest, Stein.

Let $G = (V, E)$ be a flow network with source s, sink t, and an integer capacity $c(u, v)$ on each edge $(u, v) \in E$. Let $C = \max_{(u,v)\in E} c(u, v)$ be the maximum capacity.

1. Argue that a minimum cut of G has capacity at most $C \cdot |E|$.

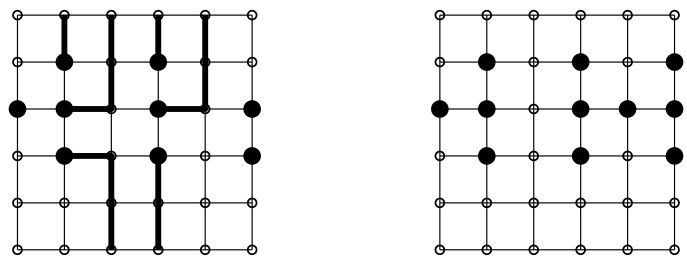

Figure 1: Grids for the escape problem. Starting points are black; other grid vertices are small circles. The first grid is shown with escapes; the second has no escape

2. For a given number K, show that an augmenting path of capacity at least K can be found in $O(|E|)$ time if such a path exists.
3. The following modification of the Ford-Fulkerson algorithm can be used to compute a maximum flow in G.

   $C := \max_{(u,v)\in E} c(u,v)$
   initialize a flow f to be 0 for each edge;
   $K := 2^{\lfloor \log C \rfloor}$
   while $K \geq 1$ **do**
       while there exists an augmenting path p of capacity at least K **do**
           augment flow f along p
       enddo
       $K := K/2$;
   enddo

4. Argue that the algorithm given above returns a maximum flow.
5. Show that the capacity of a minimum cut of the residual graph G_f is at most $2K \cdot |E|$ each time the fourth line (while $K \geq 1$) is executed.
6. Argue that the inner while loop of the 5th and 6th line is executed $O(E)$ times for each value of K.
7. Conclude that the algorithm can be implemented such that it runs in $O(|E|^2 \cdot \log C)$ time. x

4. Plugs for oil-pipes. 2 points One of my colleagues has a parent that works at a Russian oil-company. This company has the following problem. Given is an undirected graph $G = (V, E)$ with two special vertices, s, t. The edges in the graph model pipes through which oil is going. Each of these pipes/edges $e \in E$ has a *size* $s(e) \in \mathbf{N}$.

The company can place k plugs: each plug can be placed at one vertex in V.

There is the possibility of a *leakage*. A leakage occurs at an edge. If there is a leakage at edge e, the company loses all oil at edge e, and all oil in edges, connected to e via paths without plugs. (What happens is that the plugs close the pipes, and prevent the oil to flow through it, thus preventing further leaking of oil.)

The problem is to find a place to put the plugs, such that the maximum leakage that is possible when one pipe leaks is minimized.

I.e., we want to find a set of k vertices W, such that the maximum total size $s(F)$ of sets of edges F that are connected via vertices not in W is as small as possible.

Figure 2: A graph that is a path

1. Give an algorithm that solves this problem efficiently when G is a path. Analyze the running time of your algorithm. (Paths are trees where every vertex has degree one or two. See for an example the figure.)

2. **Bonus question:** Show that the problem is NP-hard on arbitrary graphs.

Bonus question: energy constrained flow Note: the bonus question is optional; not very easy, but can give you bonus points, with a maximum of a note of 13.

The following problem models a question from wireless sensor networks.

We have a directed acyclic graph $G = (V, A)$ and a source s and a sink t. Each arc $a \in A$ has a cost $c(a) \in \mathbf{Z}^+$. Each vertex $v \in V$ has an amount of energy $E(v) \in \mathbf{Z}^+$. Sending a message from v to w across arc $a = (v, w)$ costs $c(a)$ energy, which has to be spend by v from his amount $E(v)$.

A flow respecting energies is a function $f : A \to R^+ \cup \{0\}$, such that

- We have flow conservation for vertices, except s and t:

$$\forall v \in V - \{s,t\} : \sum_{(w,v)\in A} f((w,v)) = \sum_{(v,w)\in A} f((v,w))$$

- Each vertex has the energy to send its messages:

$$\forall v \in V : \sum_{(v,w)\in A} f((v,w)) \cdot c((v,w)) \leq E(v)$$

The value of a flow respecting energies f is

$$\sum_{(s,v)\in A} f((s,v))$$

1. Bonus question: Show that we can find in polynomial time a flow respecting energies with a maximum value.

2. Bonus question: suppose that we want that all values $f((v,w))$ are integers. (The problem models the sending of messages, and we assume we cannot send messages partially.) Show that the problem to find a flow respecting energies with a maximum value becomes NP-hard when we require that the flow is integral.