Exercises Algorithms and Networks – Set 6 - 2012/2013

As base for NP-completeness proofs, you may use results from the course, or results from other exercises, even when you do not prove them yourself to be NP-hard.

The first of the questions 2 – 8 that you make correctly gives 2 points; then each other correct question gives one point. The bonus question is quite difficult, can give you a note higher than 10, and gives 2 points.

Deadline: October 29, 2012.

1. NP-completeness of Clique (2 points)

(In this first exercise, the different steps in the proof are asked explicitly. Other NP-completeness proof have the same structure, but you have to fill in details yourself ...)

We consider the following problem.

CLIQUE
Instance: Undirected graph $G = (V, E)$, integer K.
Question: Does G contain a clique of at least K vertices, i.e., is there is a subset W with $|W| \geq k$, and W forms a clique?

(a) Show that CLIQUE belongs to the class NP.

(b) Let $G = (V, E)$ be a graph. The complement of G is the graph $G^c = (V, E^c)$ with $E^c = \{\{v, w\} \mid v, w \in V,\ v \neq w,\ \{v, w\} \notin E\}$, i.e., a pair of disjunct vertices is adjacent in G^c, if and only if they are not adjacent in G. Suppose W is an independent set in G. Show that W is a clique in G^c.

(c) Proof that G has an independent set of size at least K, if and only if G^c has a clique of size at least K. (Hint: you must show two things: if G has an independent set of size at least K, then G^c has a clique of size at least K, and if G^c has a clique of size at least K, then G has an independent set of size at least K¿

(d) Argue that INDEPENDENT SET is polynomial-time Karp reducible to CLIQUE.

(e) Argue that CLIQUE is NP-complete.

2. NP-completeness of Vertex Cover

A set of vertices $W \subseteq V$ is a *vertex cover* in a graph $G = (V, E)$, if for each $\{v, w\} \in E$, $v \in W$ or $w \in W$. VERTEX COVER is the following well known problem.

> VERTEX COVER
> **Instance:** Undirected graph $G = (V, E)$, integer K.
> **Question:** Does G have a vertex cover of at most K vertices?

(a) Show that for each set of vertices $W \subseteq V$ and graph $G = (V, E)$, we have that W is a vertex cover, if and only if $V - W$ is an independent set.

(b) Proof that the VERTEX COVER problem is NP-complete.

3. NP-completness of Feedback Vertex Set

A feedback vertex set in a graph $G = (V, E)$ is a set of vertices $W \subseteq V$, such that each cycle in G contains at least one vertex in W.

> FEEDBACK VERTEX SET
> **Instance:** Undirected graph $G = (V, E)$, integer K.
> **Question:** Does G contain a feedback vertex set of at most K vertices?

Show that FEEDBACK VERTEX SET is NP-complete. (Hint: what happens when we add for each edge $\{v, w\}$ in G one new vertex of degree two, with an edge to v and an edge to w?)

4. NP-completeness of Weighted Feedback Vertex Set

Show that the following problem is NP-complete.

> WEIGHTED FEEDBACK VERTEX SET
> **Instance:** Undirected graph $G = (V, E)$, integer K, weight function $w : V \rightarrow \mathbf{N}$.
> **Question:** Does G contain a feedback vertex set of total weight at most K?

I.e., we look for a set of vertices W such that each cycle in G contains at least one vertex in W and $\sum_{v \in W} w(v) \leq K$.

5. An NP-completeness proof for Longest Cycle

Show that the following problem is NP-complete.

> LONGEST CYCLE
> **Instance:** Undirected graph $G = (V, E)$, integer K.
> **Question:** Does G have a simple cycle with length at least K?

6. Task scheduling

We consider the following scheduling problem. There is one machine and a set of n tasks, $a_1, \ldots, a_n$. Each task a_i ($1 \le i \le n$) has a processing time t_j, a profit p_j, and a deadline d_j. We must schedule the tasks on the machine, such that the machine carries out at each moment at most one task; tasks run without interruption for t_j time. Tasks that are complete before their deadline give a profit of p_j; other tasks give a profit 0. Suppose a target profit P is given. Show that the problem to decide if a schedule with profit at least P is NP-complete.

7. Parcels and two trucks

A company has two trucks, and must deliver a number of parcels to a number of addresses. They want both drivers to be home at the end of the day. This gives the following decision problem.

> **Instance:** Set V of locations, with for each pair of locations $v, w \in V$, a distance $d(v, w) \in \mathbf{N}$, a starting location $s \in V$, and an integer K.
> **Question:** Are there two cycles, that both start in s, such that every location in V is on at least one of the two cycles, and both cycles have length at most K?

Show that this problem is NP-complete.

8. NP-completeness of Special 3-SAT

Recall that a literal is a variable (e.g., x_1) or its negation (**not** x_1).

> SPECIAL 3-SAT
> **Instance**: set of variables $x_1, \ldots, x_n$, collection of clauses, each with at most three literals.
> **Question:** Does there exist a truth assignment to the variables, such that each clause contains **exactly one** true literal.

Show that this problem is NP-complete. (Hint: use 3-sat. You may need to introduce additional variables.)

Bonus question

Show that the following problem is NP-complete.

> **Instance:** A planar graph $G = (V, E)$, integer K.
> **Question:** Is there a function $f : V \to \{1, 2, 3, 4\}$, such that for all $\{v, w\} \in E$: $f(v) \neq f(w)$ and $\sum_{v \in V} f(v) \leq K$.

Remark: for this bonus question, you are allowed to use the fact that the following problem is NP-complete. Also, if you find in the literature other problems that are NP-complete and you want to use, than that is fine too.

> 3-COLORABILITY OF PLANAR GRAPHS
> **Instance:** A planar graph $G = (V, E)$.
> **Question:** Is there a function $f : V \to \{1, 2, 3\}$, such that for all $\{v, w\} \in E$: $f(v) \neq f(w)$?