Shortest paths

Algorithms and Networks

Universiteit Utrecht

Contents

- The shortest path problem:
  - Statement
  - Versions
- Applications
- Algorithms (for *single source* sp problem)
  - Reminders: relaxation, Dijkstra, Variants of Dijkstra, Bellman-Ford, Johnson …
  - Scaling technique (Gabow's algorithm)
- Variant algorithms: A*, bidirectional search
- Bottleneck shortest paths

Notation

- In the entire course:
  - n = |V|, the number of vertices
  - m = |E| or m = |A|, the number of edges or the number of arcs

1

Definition and Applications

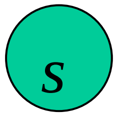
Shortest path problem

- (Directed) graph G=(V,E), length for each edge *e* in E, *w*(*e*)
- Distance from *u* to *v*: length of shortest path from *u* to *v*
- Shortest path problem: find distances, find shortest paths …

- Versions:
  - All pairs
  - Single pair
  - Single source
  - Single destination
  - Lengths can be
    - All equal (unit lengths) (BFS)
    - Non-negative
    - Negative but no negative cycles
    - Negative cycles possible

Notations

- $d_w(s,t)$: distance of s to t: length of shortest path from s to t when using edge length function w
- $d(s,t)$: the same, but w is clear from context
- $d(s,s) = 0$: we always assume there is a path with 0 edges from a vertex to itself:



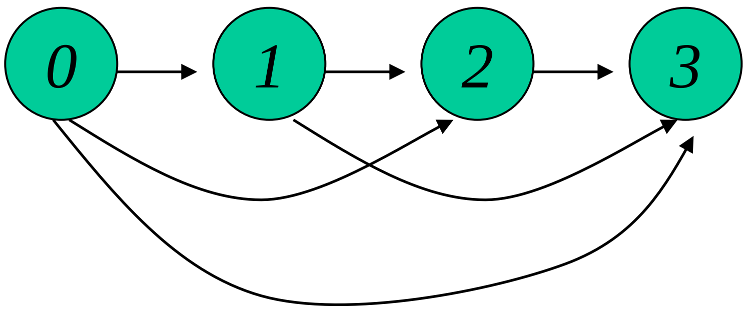
Applications

- Subroutine in other graph algorithms
- Route planning
- Difference constraints
- Allocating Inspection Effort on a Production Line

Application 1

Allocating Inspection Efforts on a Production Line

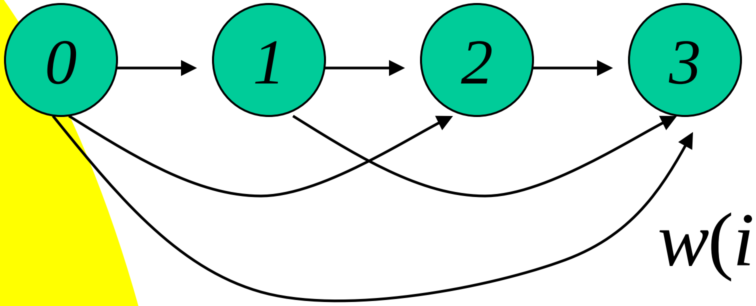
- *Production line*: ordered sequence of *n* production stages
- Each stage can make an item *defect*
- Items are inspected at some stages
- Minimize cost…



Allocating Inspection Efforts on a Production Line

- Production line: ordered sequence of n production stages.
- Items are produced in batches of B > 0 items.
- Probability that stage i produces a defect item is a_i.
- Manufacturing cost per item at stage i: p_i.
- Cost of inspecting at stage j, when last inspection has been done at stage i:
  - f_{ij} per batch, plus
  - g_{ij} per item in the batch
- When should we inspect to minimize total costs?

Solve by modeling as shortest paths problem



$$w(i,j) = f_{ij} + B(i)g_{ij} + B(i)\sum_{k=i+1}^{j} p_k$$

Where B(i) denotes the expected number of non-defective items after stage i

$$B(i) = B\prod_{k=1}^{i}(1-a_k)$$

Idea behind model

- w(i,j) is the cost of production and inspection from stage i to stage j, assuming we inspect at stage i, and then again at stage j
- Find the shortest path from 0 to n

Application 2: Difference constraints

- Tasks with precedence constraints and running length
- Each task *i* has
  - Time to complete $b_i > 0$
- Some tasks can be started after other tasks have been completed:
  - Constraint: $s_j + b_j \leq s_i$
- First task can start at time 0. When can we finish last task?
- Shortest paths problem on directed acyclic graphs (see next dias)!

Model

- Take vertex for each task
- Take special vertex v_0
- Vertex v_0 models time 0
- Arc (v_0, i) for each task vertex i, with length 0
- For each precedence constraint $s_j + b_j \leq s_i$ an arc (j, i) with length b_j

Long paths give time lower bounds

- If there is a path from v_0 to vertex i with length x, then task i cannot start before time x
- Proof with induction...
- Optimal: start each task i at time equal to length of longest path from v_0 to i.
  - *This gives a valid scheme, and it is optimal by the solution*

Difference constraints as shortest paths

- The longest path problem can be solved in O(n+m) time, as we have a directed acyclic graph.
- Transforming to shortest paths problem: multiply all lengths and times by –1.

2

Algorithms for shortest path problems
(reminders)

Basis of single source algorithms

- Source s.
- Each vertex v has variable $D[v]$
  - Invariant: $d(s,v) \leq D[v]$ for all v
  - Initially: $D[s]=0$; $v \neq s$: $D[v] = \infty$
- Relaxation step over edge (u,v):
  - $D[v] = \min \{ D[v], D[u]+ w(u,v) \}$

Maintaining shortest paths

- Each vertex maintains a pointer to the `previous vertex on the current shortest path' (sometimes NIL): p(v)
- Initially: p(v) = NIL for each v
- Relaxation step becomes:

Relax (u,v,w)

If D[v] > D[u]+ w(u,v)
  then D[v] = D[u] + $w(u,v)$; p(v) = u

p-values build paths of length D(v)
Shortest paths tree

Dijkstra

- Initialize
- Take priority queue Q, initially containing all vertices
- While Q is not empty,
  - Select vertex *v* from Q of minimum value D[*v*]
  - Relax across all outgoing edges from *v*
    - *Note*: *each relaxation can cause a change of a D-value and thus a change in the priority queue*
    - *This happens at most |E| times*

On Dijkstra

- Assumes all lengths are non-negative
- Correctness proof (done in `Algoritmiek')
- Running time:
  - Depends on implementation of priority queue
    - O(n^2): standard queue
    - O($m + n \log n$): Fibonacci heaps
    - O(($m + n$) $\log n$): red-black trees, heaps
    - …

Negative lengths

- What if w(u,v) < 0?
- Negative cycles, reachable from s …
- Bellman-Ford algorithm:
  – For instances without negative cycles:
    - In O(nm) time: SSSP problem when no negative cycles reachable from s
    - Also: detects negative cycle

Bellman-Ford

Clearly:
$O(nm)$ time

- Initialize
- Repeat |V|-1 times:
  - For every edge (u,v) in E do: Relax(u,v,w)
- For every edge (u,v) in E do

  If D[v] > D[u] + w(u,v)

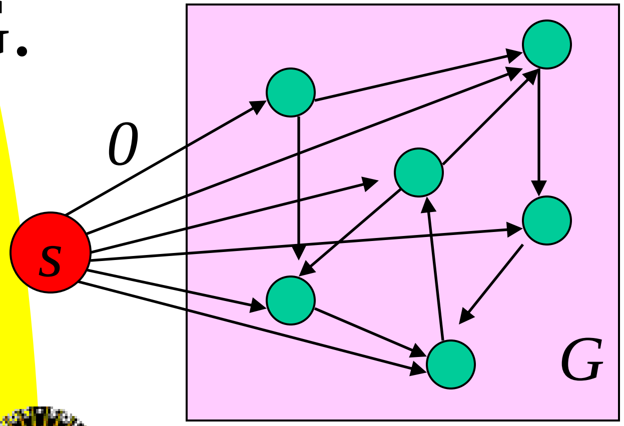
  then There exists a negative circuit! Stop
- There is no negative circuit, and for all vertices v: D[v] = d(s,v).

Correctness of Bellman-Ford

- Invariant: If no negative cycle is reachable from s, then after i runs of main loop, we have:
  - If there is a shortest path from s to u with at most i edges, then D[u]=d[s,u], for all u.
- If no negative cycle reachable from s, then every vertex has a shortest path with at most n – 1 edges.
- If a negative cycle reachable from s, then there will always be an edge with a relaxation possible.

Finding a negative cycle in a graph

- Reachable from *s*:
  - Apply Bellman-Ford, and look back with pointers
- Or: add a vertex *s* with edges to each vertex in G.



All pairs

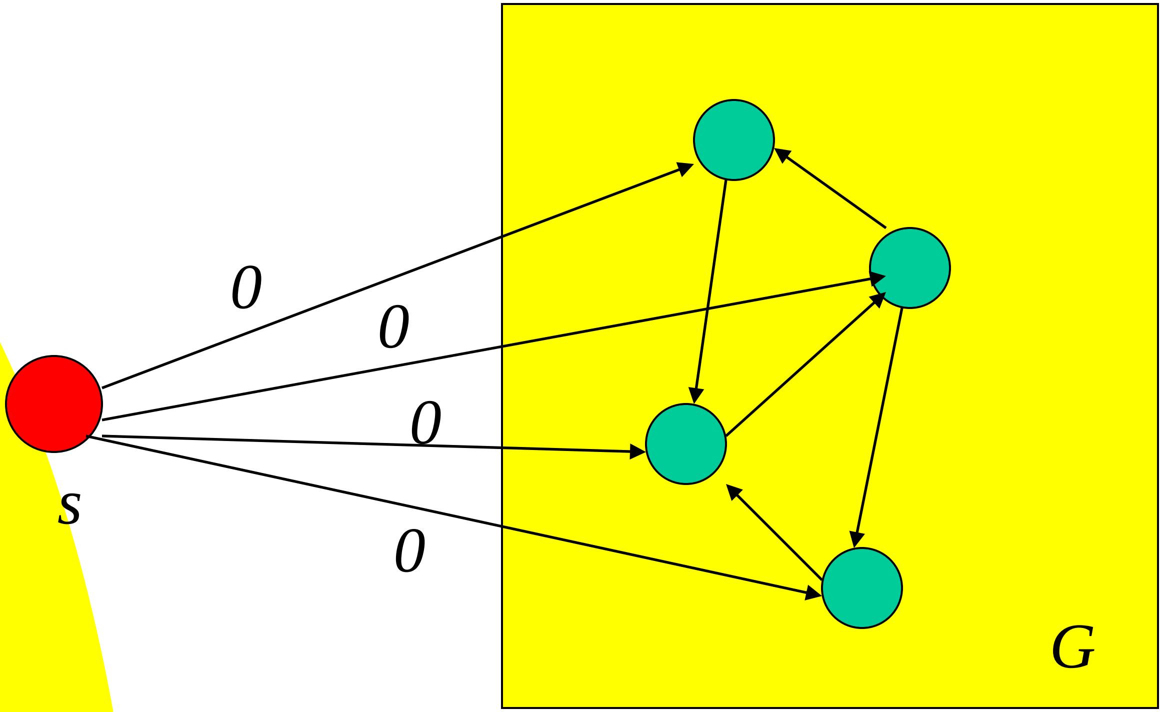
- Dynamic programming: O(n^3) (Floyd, 1962)
- Johnson: improvement for sparse graphs with reweighting technique:
  - O($n^2 \log n + nm$) time.
  - Works if no negative cycles
  - Observation: if all weights are non-negative we can run Dijkstra with each vertex as starting vertex: that gives O($n^2 \log n + nm$) time.
  - What if we have negative lengths: reweighting…

Reweighting

- Let h: V $\rightarrow$ R be any function to the reals.
- Write $\mathbf{w_h(\mathit{u},\mathit{v}) = w(\mathit{u},\mathit{v}) + h(\mathit{u}) - h(\mathit{v})}$.
- Lemmas:
  - Let P be a path from x to y. Then: $w_h(P) = w(P) + h(x) - h(y)$.
  - $d_h(x,y) = d(x,y) + h(x) - h(y)$.
  - P is a shortest path from x to y with lengths w, if and only if it is so with lengths w_h.
  - G has a negative-length circuit with lengths w, if and only if it has a negative-length circuit with lengths w_h.

What height function h is good?

- Look for height function h with
  - $w_h(u,v) \geq 0$, for all edges (u,v).
- If so, we can:
  - Compute $w_h(u,v)$ for all edges.
  - Run Dijkstra but now with $w_h(u,v)$.
- Special method to make h with a SSSP problem, and Bellman-Ford.


0
0
0
s
0
G

Choosing *h*

- Set $h(v) = d(s,v)$ (*in new graph*)
- Solving SSSP problem with negative edge lengths; use Bellman-Ford.
- If negative cycle detected: stop.
- Note: for all edges (u,v): $w_h(u,v) = w(u,v) + h(u) - h(v) = w(u,v) + d(s,u) - d(s,v) \geq 0$

Johnson's algorithm

- Build graph G' (as shown)
- Compute with Bellman-Ford $d(s,v)$ for all v
- Set $w_h(u,v) = w(u,v) + d_{G'}(s,u) - d_{G'}(s,v)$ for all edges (u,v).
- For all u do:
  - Use Dijkstra's algorithm to compute $d_h(u,v)$ for all v.
  - Set $d(u,v) = d_h(u,v) + d_{G'}(s,v) - d_{G'}(s,u)$.

$\boxed{O(n^2 \log n + nm) \text{ time}}$

3

Shortest path algorithms
"using the numbers" and scaling

Using the numbers

- Back to the single source shortest paths problem with non-negative distances
  - Suppose Δ is an upper bound on the maximum distance from s to a vertex v.
  - Let L be the largest length of an edge.
  - Single source shortest path problem is solvable in $O(m + \Delta)$ time.

In O(m+Δ) time

- Keep array of doubly linked lists: L[0], …, L[Δ],
- Maintain that for v with D[v] ≤ Δ,
  - v in L[D[v]].
- Keep a *current minimum* μ .
  - Invariant: all L[k] with $k < \mu$ are empty
- Changing D[v] from x to y: take v from L[x] (with pointer), and add it to L[y]: O(1) time each.
- Extract min: while L[μ] empty, μ++; then take the first element from list L[μ].
- Total time: O(m+Δ)

Corollary and extension

- SSSP: in O(m+nL) time. (Take Δ=nL).
- Gabow (1985): SSSP problem can be solved in O(m $\log_R$ L) time, where
  – R = max{2, m/n}
  – L : maximum length of edge
- Gabow's algorithm uses **scaling** technique!

Gabow's algorithm
Main idea

- First, build a scaled instance:
  - For each edge e set $w'(e) = \lfloor w(e) / R \rfloor$.
- Recursively, solve the scaled instance.
- Another shortest paths instance can be used to compute the correction terms!

How far are we off?

- We want d(s,v)
- R * $d_{w'}(s,v)$ is when we scale back our scaled instance: what error did we make when rounding?
- Set for each edge (x,y) in E:
  – $Z(x,y) = w(x,y) - R * d_{w'}(s,x) + R * d_{w'}(s,y)$
    - Works like height function, so the same shortest paths!
    - Height of x is $-R * d_{w'}(s,x)$

A claim

- For all vertices v in V:
  - $d(s,v) = d_Z(s,v) + R * d_{w'}(s,v)$
- As with height functions (telescope):
  - $d(s,v) = d_Z(s,v) + h(s) – h(v) = d_Z(s,v) – R*d_{w'}(s,s) + R * d_{w'}(s,v)$
  - And $d_{w'}(s,s) = 0$
- Thus, we can compute distances for w by computing distances for Z and for w'

Gabow's algorithm

If L <= R, then

- solve the problem using the $O(m+nR)$ algorithm (Base case)

Else

- For each edge e: set $w'(e) = \lfloor w(e) / R \rfloor$.
- Recursively, compute the distances *but with the new length function* w'. Set for each edge (u,v):
  - $Z(u,v) = w(u,v) + R * d_{w'}(s,u) - R * d_{w'}(s,v)$.
- Compute $d_Z(s,v)$ for all v (how? See next!) and then use
  - $d(s,v) = d_Z(s,v) + R * d_{w'}(s,v)$

A Property of Z

- For each edge $(u,v) \in$ E we have:
  - $Z(u,v) = w(u,v) + R^* \, d_{w'}(s,u) - R * d_{w'}(s,v) \geq 0$, because
  - $w(u,v) \geq R * w'(u,v) \geq R * (d_{w'}(s,v) - d_{w'}(s,u))$.
- So, a variant of Dijkstra can be used to compute distances for Z.

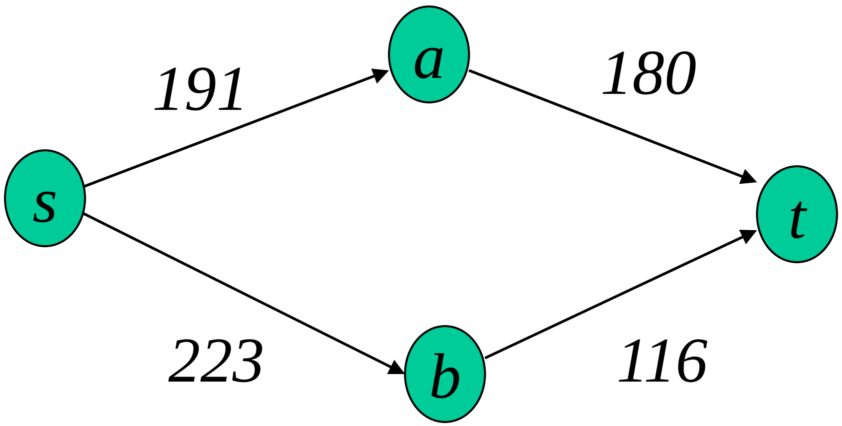
Computing distances for Z

- For each vertex *v* we have
  - $d_Z(s,v) \le n$R for all *v* reachable from *s*
    - Consider a shortest path P for distance function *w*' from *s* to *v*
    - For each of the less than *n* edges *e* on P, $w(e) \le R + R*w'(e)$
    - So, $d(s,v) \le w(P) \le nR + R*w'(P) = nR + R*d_{w'}(s,v)$
    - Use that $d(s,v) = d_Z(s,v) + R * d_{w'}(s,v)$
- So, we can use $O(m+nR)$ algorithm (Dijkstra with doubly-linked lists) to compute all values $d_Z(v)$.

Gabow's algorithm (analysis)

- Recursive step: costs O(m $\log_R L'$) with $L' = \lfloor L/R \rfloor$.
- SSSP for Z costs $O(m + nR) = O(m)$.
- Note: $\log_R L' \leq (\log_R L) - 1$.
- So, Gabow's algorithm uses $O(m \log_R L)$ time.

Example



4

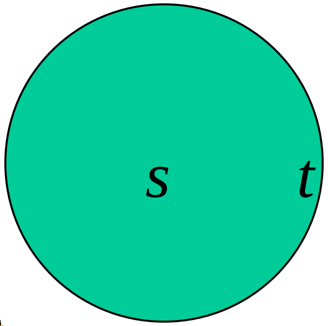
Variants: A* and bidirectional search

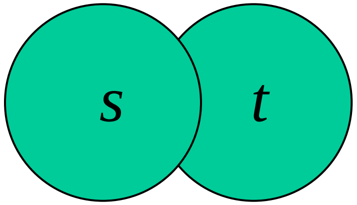
A* (simplified exposition)

- Practical importance: A* algorithm
- Can be explained in terms of height functions
- Consider a route planning problem with geographic data, and distances of arcs at least Euclidian distance of vertices
- Suppose we have a single pair shortest path problem, with source *s* and target *t*
- Use height function - h(*v*) = - the Euclidian distance from *v* to *t* (notate: $|vt|$)
- For all arcs $(v,w) \in E$:
  - $w'(v,w) = w(v,w) - |vt| + |wt| \geq |vw| - |vt| + |wt| \geq 0$
- Note: arcs towards target get smaller length and arcs away from target larger length
- Algorithm is faster in practice but still correct

Bidirectional search

- For a single pair shortest path problem:
- Start a Dijkstra-search from both sides simultaneously
- Analysis needed for stopping criterion
- Faster in practice
- Combines nicely with A*





5

Bottleneck shortest paths

Bottleneck shortest path

- Given: weighted graph G, weight w(*e*) for each arc, vertices *s*, *t*.
- Problem: find a path from *s* to *t* such that the **maximum weight** of an arc on the path is as small as possible.
  - Or, reverse: such that the minimum weight is as large as possible: *maximum capacity path*

Algorithms

- On directed graphs: O((m+n) log m),
  - Or: O((m+n) log L) with L the maximum absolute value of the weights
  - Binary search and DFS
- On undirected graphs: O(m+n) with divide and conquer strategy

Bottleneck shortest paths on undirected graphs

- Find the median weight of all weights of edges, say *r*.
- Look to graph G^r formed by edges with weight at most *r*.
- If *s* and *t* in same connected component of G^r, then the bottleneck is at most *r*: now remove all edges with weight more than *r*, and repeat (recursion).
- If *s* and *t* in different connected components of G^r: the bottleneck is larger than *r*. Now, contract all edges with weight at most *r*, and recurse.
- $T(m) = O(m) + T(m/2)$

5

Conclusions

Conclusions

- Applications
- Several algorithms for shortest paths
  - Variants of the problem
  - Detection of negative cycles
  - Reweighting technique
  - Scaling technique
- A*, bidirectional
- Bottleneck shortest paths