The Traveling Salesperson Problem

Algorithms and Networks

Universiteit Utrecht

Contents

- TSP and its applications
- Heuristics and approximation algorithms
  - Construction heuristics, a.o.: Christofides, insertion heuristics
  - Improvement heuristics, a.o.: 2-opt, 3-opt, Lin-Kernighan

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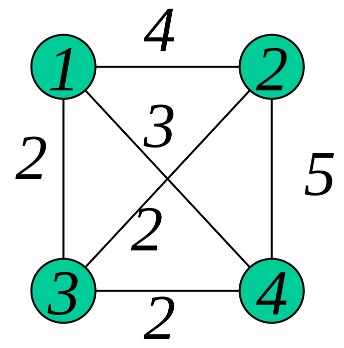
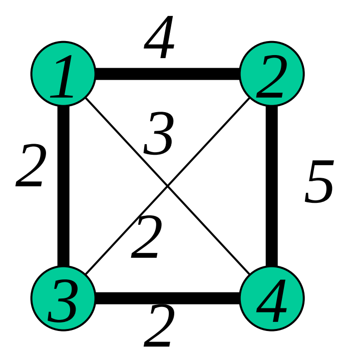
Problem definition

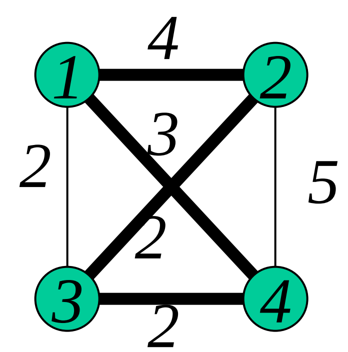
Applications

Problem

- *Instance*: *n* vertices (cities), distance between every pair of vertices
- *Question*: Find shortest (simple) cycle that visits every city







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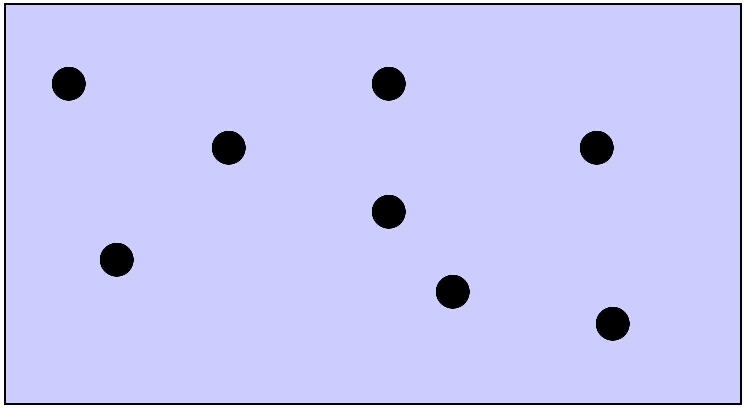
Applications

- Collection and delivery problems
- Robotics
- Board drilling



NP-complete

- *Instance*: cities, distances, *K*
- *Question*: is there a TSP-tour of length at most *K*?
  - Is an NP-complete problem
  - Relation with Hamiltonian Circuit problem

Assumptions

- Lengths are non-negative (or positive)
- Symmetric: $w(u,v) = w(v,u)$
  - Not always: painting machine application
- Triangle inequality: for all x, y, z:

  $\mathbf{w(x,y) + w(y,z) \geq w(x,z)}$
- Always valid?

If triangle inequality does not hold

Theorem: If P≠ NP, then there is no polynomial time algorithm for TSP without triangle inequality that approximates within a ratio *c*, for any constant *c*.

Proof: Suppose there is such an algorithm A. We build a polynomial time algorithm for Hamiltonian Circuit (giving a contradiction):

- Take instance G=(V,E) of HC
- Build instance of TSP:
  - A city for each $v \in V$
  - If $(v,w) \in E$, then $d(v,w) = 1$, otherwise $d(v,w) = nc+1$
- A finds a tour with distance at most nc, if and only if G has a Hamiltonian circuit

Heuristics and approximations

- Two types
  - Construction heuristics
    - A tour is built from nothing
  - Improvement heuristics
    - Start with `some' tour, and continue to change it into a better one as long as possible

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Construction heuristics

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1^{st} Construction heuristic: Nearest neighbor

- Start at some vertex s; $v=s$;
- While not all vertices visited
  - Select closest unvisited neighbor w of v
  - Go from v to w;
  - $v=w$
- Go from v to s.

Can have performance ratio $O(\log n)$

Heuristic with ratio 2

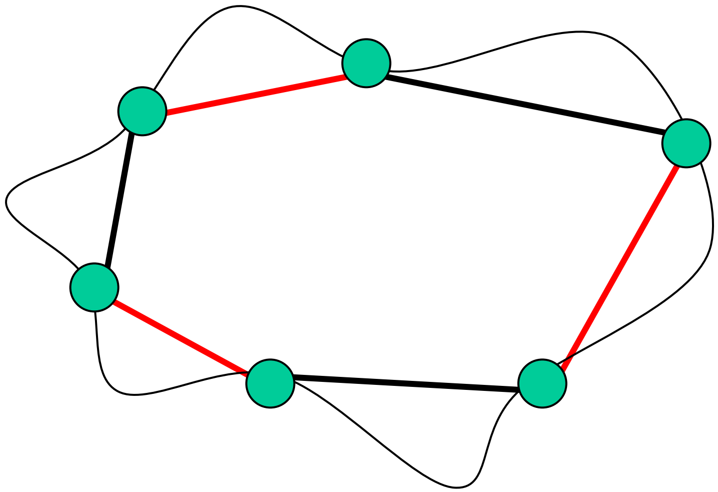
- Find a minimum spanning tree
- Report vertices of tree in preorder

Christofides

- Make a Minimum Spanning Tree T
- Set W = {v | v has odd degree in tree T}
- Compute a minimum weight matching M in the graph G[W].
- Look at the graph T+M. (Note: Eulerian!)
- Compute an Euler tour C' in T+M.
- Add shortcuts to C' to get a TSP-tour

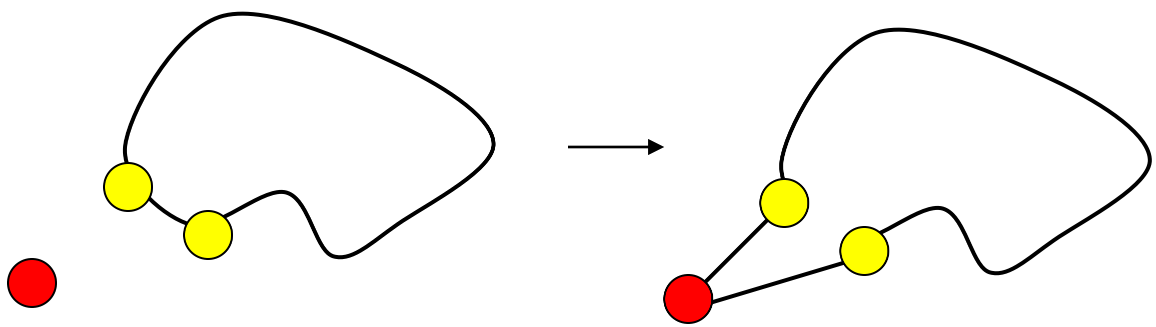
Ratio 1.5

- Total length edges in T: at most OPT
- Total length edges in matching M: at most OPT/2.
- T+M has length at most 3/2 OPT.
- Use Δ-inequality.



Closest insertion heuristic

- Build tour by starting with one vertex, and inserting vertices one by one.
- Always insert vertex that is closest to a vertex already in tour.



Closest insertion heuristic has performance ratio 2

- Build tree T: if v is added to tour, add to T edge from v to closest vertex on tour.
- T is a Minimum Spanning Tree (Prim's algorithm)
- Total length of T $\leq$ OPT
- Length of tour $\leq$ 2* length of T

Many variants

- **Closest insertion**: insert vertex closest to vertex in the tour
- **Farthest insertion**: insert vertex whose minimum distance to a node on the cycle is maximum
- **Cheapest insertion**: insert the node that can be inserted with minimum increase in cost
  - Gives also ratio 2
  - Computationally expensive
- **Random insertion**: randomly select a vertex
- *Each time: insert vertex at position that gives minimum increase of tour length*

Cycle merging heuristic

- Start with *n* cycles of length 1
- Repeat:
  - Find two cycles with minimum distance
  - Merge them into one cycle
- Until 1 cycle with *n* vertices
- This has ratio 2: compare with algorithm of Kruskal for MST.

Savings

- Cycle merging heuristic where we merge tours that provide the largest “savings”: can be merged with the smallest additional cost / largest savings

Some test results

- In an overview paper, Junger et al report on tests on set of instances (105 – 2392 vertices; city-generated TSP benchmarks)
- Nearest neighbor: 24% away from optimal in average
- Closest insertion: 20%;
- Farthest insertion: 10%;
- Cheapest insertion: 17%;
- Random Insertion: 11%
- Preorder of min spanning tress: 38%
- Christofides: 19% with improvement 11% / 10%
- Savings method: 10% (and fast)

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Improvement heuristics

Improvement heuristics

- Start with a tour (e.g., from heuristic) and improve it stepwise
  - 2-Opt
  - 3-Opt
  - K-Opt
  - Lin-Kernighan
  - Iterated LK
  - Simulated annealing, …

Iterative improvement

Local search

Scheme

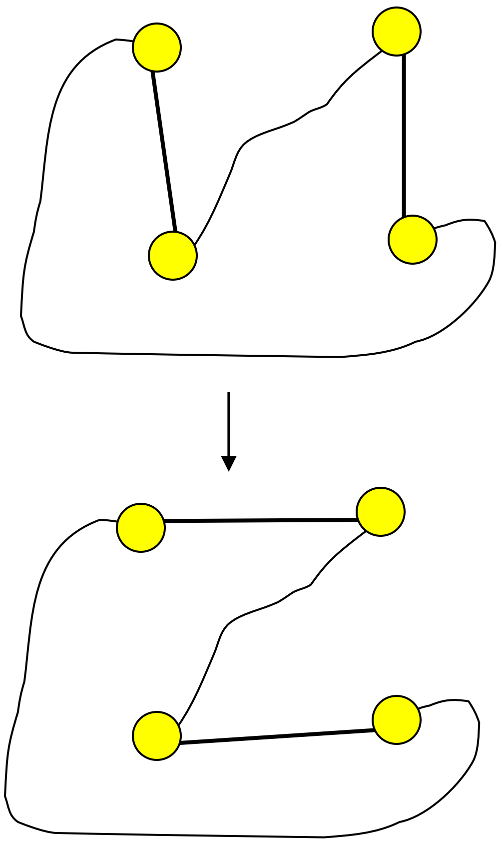
- Rule that modifies solution to different solution
- While there is a Rule(sol, sol’) with sol’ a better solution than sol
  - Take sol’ instead of sol
- Cost decrease
- Stuck in `local minimum’
- Can use exponential time in theory…

Very simple

- Node insertion
  - Take a vertex *v* and put it in a different spot in the tour
- Edge insertion
  - Take two successive vertices *v*, *w* and put these as edge somewhere else in the tour

2-opt

- Take two edges (*v*,*w*) and (*x*,*y*) and replace them by (*v*,*x*) and (*w*,*y*) OR (*v*,*y*) and (*w*,*x*) to get a tour again.
- Costly: part of tour should be turned around

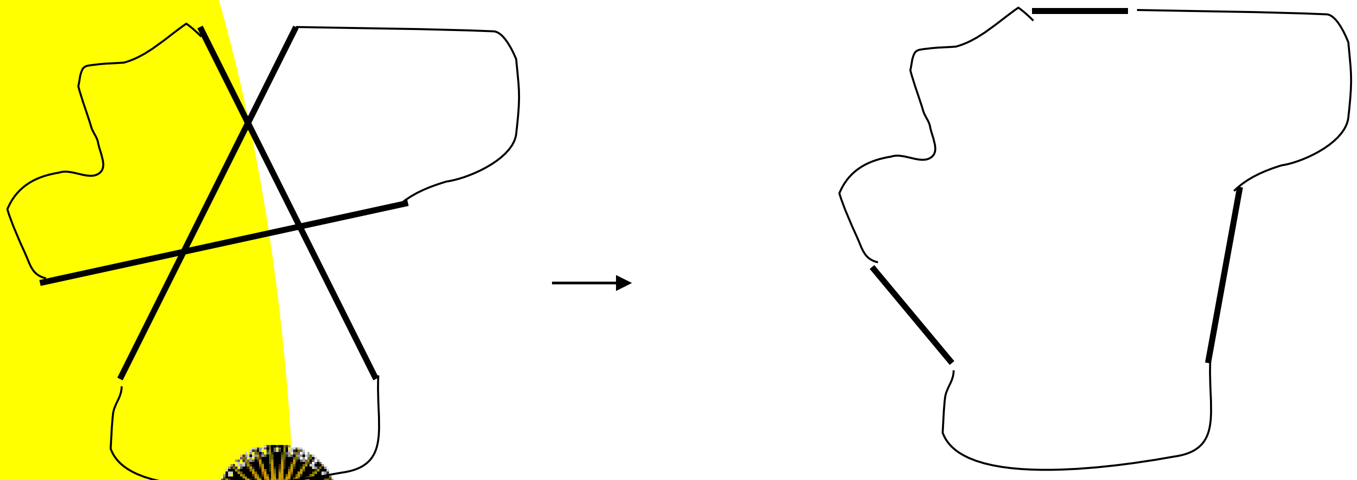


2-Opt improvements

- Reversing shorter part of the tour
- Clever search to improving moves
- Look only to subset of candidate improvements
- Postpone correcting tour
- Combine with node insertion
- On $\mathbf{R}^2$: get rid of crossings of tour

3-opt

- Choose three edges from tour
- Remove them, and combine the three parts to a tour in the cheapest way to link them



3-opt

- Costly to find 3-opt improvements: $O(n^3)$ candidates
- k-opt: generalizes 3-opt

Lin-Kernighan

- Idea: modifications that are *bad* can lead to enable something *good*
- Tour modification:
  – Collection of simple changes
  – Some increase length
  – Total set of changes decreases length

LK

- One LK step:
  - Make sets of edges X = $\{x_1, \ldots, x_r\}$, Y = $\{y_1,\ldots,y_r\}$
    - If we replace X by Y in tour then we have another tour
  - Sets are built stepwise
- Repeated until …
- Variants on scheme possible

One LK step

- Choose vertex t_1, and edge $x_1 = (t_1,t_2)$ from tour.
- i=1
- Choose edge $y_1=(t_2, t_3)$ not in tour with $g_1 = w(x_1) – w(y_1) > 0$ (or, as large as possible)
- Repeat a number of times, or until …
  - i++;
  - Choose edge $x_i = (t_{2i-1},t_{2i})$ from tour, such that
    - x_i not one of the edges y_j
    - *oldtour* – X + (t_{2i},t_1) +Y is also a tour
  - **if** *oldtour* – X + (t_{2i},t_1) +Y has shorter length than *oldtour*, **then** take this tour: done
  - Choose edge $y_i = (t_{2i}, t_{2i+1})$ such that
    - $g_i = w(x_i) – w(y_i) > 0$
    - y_i is not one of the edges x_j .
    - y_i not in the tour

Iterated LK

Cost much time
Gives excellent results

- Construct a start tour
- Repeat the following *r* times:
  - Improve the tour with Lin-Kernighan until not possible
  - Do a random 4-opt move that *does not increase the length with more than* 10 *percent*
- Report the best tour seen

Other methods

- Simulated annealing and similar methods
- Problem specific approaches, special cases
- Iterated LK combined with treewidth/branchwidth approach:
  - Run ILK a few times (e.g., 5)
  - Take graph formed by union of the 5 tours
  - Find minimum length Hamiltonian circuit in graph with clever dynamic programming algorithm

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A dynamic programming algorithm

Held-Karp algorithm for TSP

- $O(n^2 2^n)$ algorithm for TSP
- Uses Dynamic programming
- Take some starting vertex s
- For set of vertices R ($s \in R$), vertex $w \in$ R, let
  - B(R,w) = minimum length of a path, that
    - Starts in s
    - Visits all vertices in R (and no other vertices)
    - Ends in w

TSP: Recursive formulation

- B({*s*},*s*) = 0
- If |S| > 1, then
  - **$B(S,x) = \min_{v \in S - \{x\}} B(S-\{x\}, v) + w(v,x)$**
- If we have all B(V,*v*) then we can solve TSP.
- Gives requested algorithm using DP-techniques.

Conclusions

- TSP has many applications
- Also many applications for variants of TSP
- Heuristics: construction and improvement
- Further reading:
  - M. Jünger, G. Reinelt, G. Rinaldi, *The Traveling Salesman Problem*, in: *Handbooks in Operations Research and Management Science*, *volume 7: Network Models*, North-Holland Elsevier, 1995.