The Stable Marriage Problem

Algorithms and Networks

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The stable marriage problem

- Story: there are *n* men and *n* women, which are unmarried. Each has a preference list on the persons of the opposite sex
- Does there exist and can we find a stable matching (stable marriage): a matching of men and women, such that there is no pair of a man and a woman who both prefer each other above their partner in the matching?

Application

- Origin: assignment of medical students to hospitals (for internships)
  - Students list hospitals in order of preference
  - Hospitals list students in order of preference

Example

- Arie: Betty Ann Cindy
- Bert: Ann Cindy Betty
- Carl: Ann Cindy Betty
- Ann: Bert Arie Carl
- Betty: Arie Carl Bert
- Cindy: Bert Arie Carl

- Stable matching: (Arie,Betty), (Bert,Ann), (Carl,Cindy)
- Matching (Arie,Ann), (Bert,Betty), (Carl, Cindy) is not stable, e.g., Arie and Betty prefer each other above given partner
- *Blocking pair*

Remark

- “Local search” approach does not need to terminate

  SOAP-SERIES-ALGORITHM

  While there is a blocking pair

  Do Switch the blocking pair

  – Can go on for ever!

  – So, we need something else…

Result

- Gale/Stanley algorithm: finds always a stable matching
  - Input: list of men, women, and their preference list
  - Output: stable matching

The algorithm

- Fix some ordering on the men
- Repeat until everyone is matched
  - Let X be the first unmatched man in the ordering
  - Find woman Y such that Y is the most desirable woman in X’s list such that Y is unmatched, or Y is currently matched to a Z and X is more preferable to Y than Z.
  - Match X and Y; possible this turns Z to be unmatched

Questions:
Does this terminate? How fast?
Does this give a stable matching?

Termination and number of steps

- Once a woman is matched, she stays matched (her partner can change).
- When the partner of a woman changes, this is to a more preferable partner for her: at most $n - 1$ times.
- Every step, either an unmatched woman becomes matched, or a matched woman changes partner: at most n^2 steps.

Stability of final matching

- Suppose final matching is not stable.
- Take:
  - Arie is matched to Ann,
  - Bert is matched to Betty,
  - Arie prefers Betty to Ann,
  - Betty prefers Arie to Bert.
- So: Betty is before Ann in the preference list of Arie, but Arie is not matched to Betty. Two cases:
  - When Arie considers Betty, she has a partner (say) Carl preferable to Arie: Carl is also preferable to Bert, but in the algorithm woman can get only more preferable partners, contradiction.
  - When Arie considers Betty, she is free, but Arie is later replaced by someone preferable to Arie. Again, Betty can never end up with Bert.

Comments

- A stable matching exists and can be found in polynomial time
- Consider the greedy algorithm:
  – Start with any matching, and make switches when a pair prefers each other to their current partner
  – This algorithm does not need to terminate
- Controversy: the algorithm is better for the men: hospitals in the application

Man optimal stable matchings

Theorem

1. All possible executions of the Gale-Shapley algorithm give the same stable matching.
   – A proof of this follows
2. In this matching, the men have the best partner they can have in any stable matching.
3. In this matching, the women have the worst partner they can have in any stable matching.

Proof

- Suppose the algorithm gives matching M.
- Suppose there is a stable matching M' with man *m* matched to *w'* in M', and to *w* in M, with *m* preferring *w'* over *w*.
- Look at run of algorithm that produces M. *w'* has rejected *m* at some point.
- Of all such *m*, *w* and *w'*, take a triple such that the rejection of *m* by *w'* happens first.
- Suppose *w'* prefers *m'* to *m*, as reason for the rejection.
- *m'* must prefer *w'* to his partner in M': see next slide
- Thus *m', w'* is a *blocking pair* in M': M' not stable; contradiction.
- So, all men are matched to the woman that appears in a stable matching that they prefer most.
  - Unique solution

m' Prefers *w'*

- *m'* is matched in M' with *w''*
- If *m'* prefers *w''* to *w'*:
  - In execution of algorithm, we have
    - At some point *w''* must reject *m'* as later *m'* is matched with *w'* (while *w'* rejects *m* at that step).
    - This is earlier than the rejection of *w'* of *m*
  - Now *m'*, *w'* and *w''* form an earlier choice for the triple.

Stable roommates

- Variant of problem with boys that must share two-person rooms (US campus)
- Each has preference list
- Stable marriage problem is special case

Not always a stable matching for the stable roommates

- Consider the following instance:
  - Person Arie: Carl Bert Dirk
  - Person Bert: Arie Carl Dirk
  - Person Carl: Bert Arie Dirk
  - Person Dirk: no difference
- Each matching is unstable e.g., (Arie,Bert)(Carl,Dirk) has {Carl,Arie} as blocking pair

Testing stable roommates

- Complicated algorithm
- Uses $O(n^2)$ time

Comments

- Much further work has been done, e.g.:
  – Random / Fair stable matchings
  – Many variants of stable matching are also solvable (indifferences, groups, forbidden pairs, …)
  – What happens if some participants lie about their preferences?
  – Stable roommates with indifferences: NP-complete