Fixed Parameter Complexity

Algorithms and Networks

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Fixed parameter complexity

- Analysis what happens to problem when some parameter is *small*
- Definitions
- Fixed parameter tractability techniques
  - Branching
  - Kernelisation
  - Other techniques

Motivation

- In many applications, some number can be assumed to be *small*
  - Time of algorithm can be exponential in this small number, but should be polynomial in *usual* size of problem

Parameterized problem

- Given: Graph G, integer k, ...
- Parameter: k
- Question: Does G have a ??? of size at least (at most) k?
  - Examples: vertex cover, independent set, coloring, …

Examples of parameterized problems (1)

Graph Coloring

Given: Graph G, integer k

Parameter: k

Question: Is there a vertex coloring of G with k colors? (I.e., $c: V \rightarrow \{1, 2, \ldots, k\}$ with for all $\{v,w\} \in E$: $c(v) \neq c(w)$?)

- NP-complete, even when $k=3$.

Examples of parameterized problems (2)

Clique

Given: Graph G, integer k

Parameter: k

Question: Is there a clique in G of size at least k?

- Solvable in $O(n^k)$ time with simple algorithm. Complicated algorithm gives $O(n^{2k/3})$. Seems to require $\Omega(n^{f(k)})$ time…

Examples of parameterized problems (3)

Vertex cover

Given: Graph G, integer k

Parameter: k

Question: Is there a vertex cover of G of size at most k?

- Solvable in $O(2^k (n+m))$ time

Fixed parameter complexity theory

- To distinguish between behavior:
  - $O(\ f(k) * n^c)$
  - $\Omega(\ n^{f(k)})$
- Proposed by Downey and Fellows.

Parameterized problems

- Instances of the form (*x*,*k*)
  - I.e., we have a *second parameter*
- Decision problem (subset of $\{0,1\}^* \times \mathbf{N}$)

Fixed parameter tractable problems

- FPT is the class of problems with an algorithm that solves instances of the form (x,k) in time $p(|x|)*f(k)$, for polynomial p and some function f.

Hard problems

- Complexity classes
  - **FPT** $\subseteq$ W[1] $\subseteq$ W[2] $\subseteq$ … W[i] $\subseteq$ … $\subseteq$ W[P]
  - FPT is 'easy', all others 'hard'
  - Defined in terms of *Boolean circuits*
  - Problems hard for W[1] or larger class are assumed not to be in FPT
    - Compare with P / NP

Examples of hard problems

- Clique and Independent Set are W[1]-complete
- Dominating Set is W[2]-complete
- Version of Satisfiability is W[1]-complete
  - Given: set of clauses, *k*
  - Parameter: *k*
  - Question: can we set (at most) *k* variables to **true**, and al others to **false**, and make all clauses true?

Techniques for showing fixed parameter tractability

- Branching
- Kernelisation
- Iterative compression
- Other techniques (e.g., treewidth)

A branching algorithm for vertex cover

- Idea:
  - Simple base cases
  - Branch on an edge: one of the endpoints belongs to the vertex cover
- Input: graph G and integer k

Branching algorithm for Vertex Cover

- Recursive procedure VC(Graph G, int k)
- VC(G=(V,E), k)
  - If G has no edges, then return **true**
  - If $k == 0$, then return **false**
  - Select an edge $\{v,w\} \in E$
  - Compute G' = G [V – v]
  - Compute G'' = G [V – w]
  - Return VC(G',k – 1) or VC(G'',k – 1)

Analysis of algorithm

- Correctness
  - Either v or w must belong to an optimal VC
- Time analysis
  - Recursion depth k
  - At most 2^k recursive calls
  - Each recursive call costs $O(n+m)$ time
  - $O(2^k (n+m))$ time: FPT

Cluster editing

- **Instance**: undirected graph G=(V,E), integer K
- **Parameter**: K
- **Question**: can we make at most K modifications to G, such that each connected component is a clique, where each modification is an addition of an edge or the deletion of an edge?
- Models biological question: partition species in families, where available data contains mistakes
- With branching: $O(3^k\ p(n))$ algorithm

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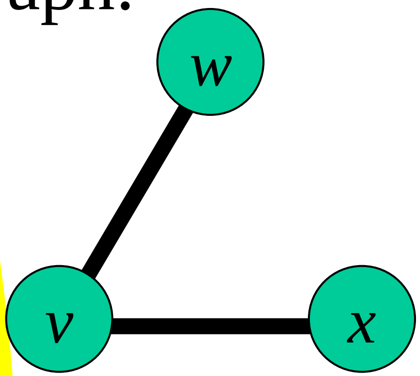
Lemma

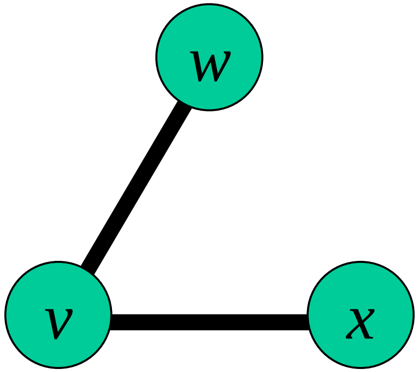
- If G has a connected component that is not a clique, then G contains the following subgraph:



- Proof: there are vertices *w* and *x* in the connected component that are not adjacent. Take such *w* and *x* of minimum distance. Case analysis: distance is 2 hence this subgraph

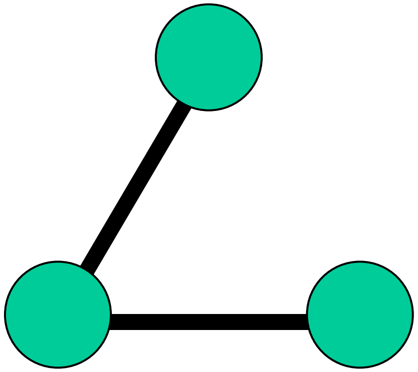
Branching algorithm for Cluster Editing

- If each connected component is a clique:
  - Answer YES
- If k=0 and some connected components are not cliques:
  - Answer NO
- Otherwise, there must be vertices v, w, x with $\{v,w\} \in E$, $\{v,x\} \in E$, and $\{w,x\} \notin E$
  - Go three times in recursion:
    - Once with $\{v,w\}$ removed and $k = k - 1$
    - Once with $\{v,x\}$ removed and $k = k - 1$
    - Once with $\{w,x\}$ added and $k = k - 1$



Analysis branching algorithm

- Correctness by lemma
- Time …
  - 3^k leaves of decision tree



More on cluster editing

- Faster branching algorithms exist
- Important applications and practical experiments
- We’ll see more when discussing kernelisation

Max SAT

- Variant of satisfiability, but now we ask: can we satisfy at least *k* clauses?
- NP-complete
- With *k* as parameter: FPT
- Branching:
  - Take a variable
  - If it only appears positively, or negatively, then …
  - Otherwise: Branch! What happens with *k*?

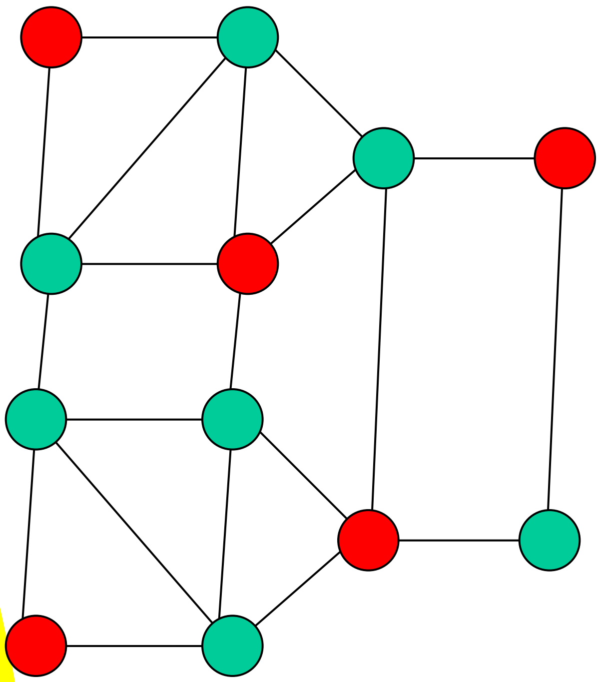
Independent Set on planar graphs

Given: a planar graph G=(V,E), integer k

Parameter: k

Question: Does G have an independent set with at least k vertices, i.e., a set W of size at least k with for all v, $w \in$ V: $\{v,w\} \notin$ E

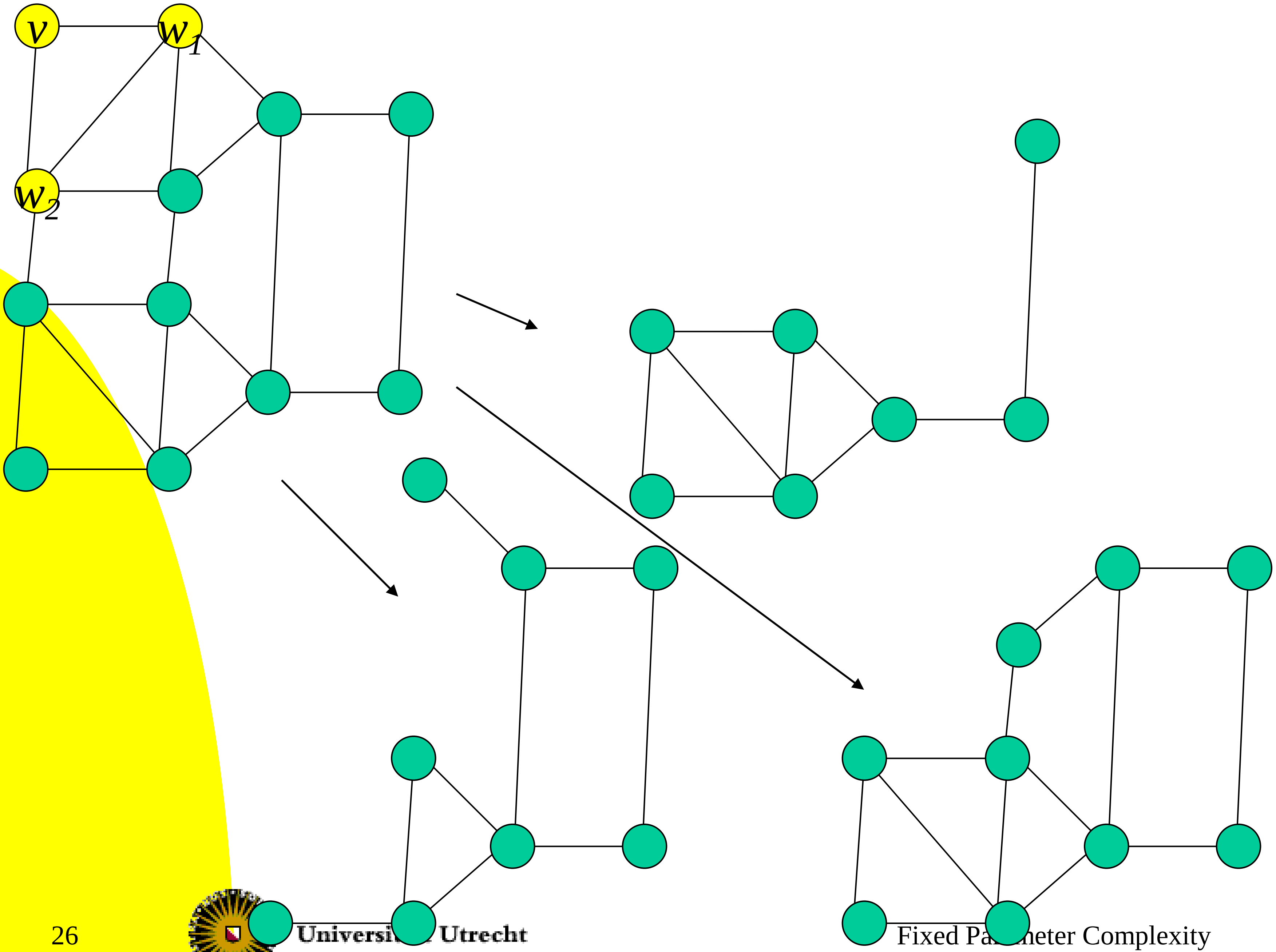
- NP-complete
- Easy to see that it is FPT by kernelisation…
- Here: $O(6^k\, n)$ algorithm



The red vertices
form an independent set

Branching

- Each planar graph has a vertex of degree at most 5
- Take vertex v of minimum degree, say with neighbors $w_1, \ldots, w_r$, r at most 5
- A maximum size independent set contains v or one of its neighbors
  - Selecting a vertex is equivalent to removing it and its neighbors and decreasing k by one
- Create at most 6 subproblems, one for each $x \in \{v, w_1, \ldots, w_r\}$. In each, we set $k = k - 1$, and remove x and its neighbors


v
w1
w2

Closest string

Given: k strings $s_1, \ldots, s_k$ each of length L, integer d

Parameter: d

Question: is there a string s with Hamming distance at most d to each of $s_1, \ldots, s_k$

- Application in molecular biology
- Here: FPT algorithm
- (Gramm and Niedermeier, 2002)

Subproblems

- Subproblems have form
  - Candidate string s
  - Additional parameter r
  - We look for a solution to original problem, with additional condition:
    - Hamming distance at most r to s
- Start with $s = s_1$ and $r=d$ (= original problem)

Branching step

- Choose an s_j with Hamming distance > d to s
- If Hamming distance of s_j to s is larger than $d+r$: *NO*
- For all positions i where s_j differs from s
  - Solve subproblem with
    - s changed at position i to value $s_j(i)$
    - $r = r - 1$
- Note: we find a solution, if and only one of these subproblems has a solution

Example

- Strings 01112, 02223, 01221, d=3
  - First position in solution will be a 0
  - First subproblem (01112, 3)
  - Creates three subproblems
    - (02113, 2)
    - (01213, 2)
    - (01123, 2)

Time analysis

- Recursion depth d
- At each level, we branch at most at $d + r \leq 2d$ positions
- So, number of recursive steps at most $2d^{d+1}$
- Each step can be done in polynomial time: $O(kdL)$
- Total time is $O(2d^{d+1} \cdot kdL)$
- Speed up possible by more clever branching and by kernelisation

More clever branching

- Choose an s_j with Hamming distance > d to s
- If Hamming distance of s_i to s is larger than $d+r$: *NO*
- Choose arbitrarily $d+1$ positions where s_j differs from s
  - Solve subproblem with
    - s changed at position i to value s_j (j)
    - $r = r - 1$
- Note: still correct, and running time can be made $O(kL + kd\, d^d)$

Technique

- Try to find a branching rule that
  - Decreases the parameter
  - Splits in a bounded number of subcases
    - YES, if and only if YES in at least one subcase

Kernelisation

- Preprocessing rules reduce starting instance to one of size f(k)
  - Should work in polynomial time
- Then use any algorithm to solve problem on kernel
- Time will be p(n) + g(f(k))

Kernelization

- Helps to analyze preprocessing
- Much recent research
- Today: definition and some examples

Formal definition of kernelisation

- Let P be a parameterized problem. (Each input of the form (I,k).)
  A *reduction to a problem kernel* is an algorithm A, that transforms inputs of P to inputs of P, such that
  - (I,k) $\in$ P, if and only if A(I,k) $\in$ P for all (I,k)
  - If A(I,k) = (I',k'), then $k' \leq$ f(k), and |I'| $\leq$ g(k) for some functions f, g
  - A uses time, polynomial in |I| and k

Kernels and FPT

- **Theorem**. Consider a decidable parameterized problem. Then the problem belongs to FPT, if and only if it has a kernel
- <= Build the kernel and then solve the problem on the kernel
- => Suppose we have an $f(k)n^c$ algorithm. Run the algorithm for n^{c+1} steps. If it did not yet solve the problem, return the input as kernel: it has size at most f(k). If it solved the problem, then …

Consequence

- If a problem is W[1]-hard, it has no kernel, unless FPT=W[1]
- There are also techniques to give evidence that problems have no kernels of polynomial size
  - If problem is *compositional* and NP-hard, then it has no polynomial kernel
  - Example is e.g., LONG PATH

First kernel: Convex string recoloring

- Application from molecular biology
- Given: string *s* in Σ^*, integer *k*
- Parameter: *k*
- Question: can we change at most *k* characters in the string *s*, such that *s* becomes *convex*, i.e., for each symbol, the positions with that symbol are consecutive.
- Example of convex string: aaacccbxxxffff
- Example of string that is not convex: abba
- Instead of symbols, we talk about *colors*

Kernel for convex string recoloring

- Theorem: Convex string recoloring has a kernel with $O(k^2)$ characters.

Notions

- Notion: good and bad colors
- A color is *good*, if it is consecutive in *s*, otherwise it is bad
- abba: a is bad and b is good
- Notion: block: consecutive occurrences of the same color: aaabbbaccc has four blocks
- Convex: each color has one block

Stepwise construction of kernel

- Step 1: limit the number of blocks of bad colors
- Step 2: limit the number of good colors
- Step 3: limit the number of characters in *s* per block
- Step 4: count

Rule 1

- If there are more than 4*k* blocks of bad colors, say NO
  - Formally, transform to trivial NO-instance, e.g. (aba, 0)
  - Why correct?

Rule 2

- If we have two consecutive blocks of good colors, then change the color of the second block to that of the first

- E.g: abbbbcca -> abbbbbba

- Why correct?

Rule 3

- If a block has more than k+1 characters, delete all but k+1 of the block
- Correctness: a block of such a size will never be changed

Counting

- After the rules have been applied, we have at most:
  - $4k$ blocks of bad colors
  - $4k+1$ blocks of good colors: at most one between each pair of bad colors, one in front and one in the end
  - Each block has size at most $k+1$
- String has size at most $(8k+1)(k+1)$
- This can be improved by better analysis, more rules, ...

Vertex cover: observations that helps for kernelisation

- If v has degree at least $k+1$, then v belongs to each vertex cover in G of size at most k.
  - If v is not in the vertex cover, then all its neighbors are in the vertex cover.
- If all vertices have degree at most k, then a vertex cover has at least m/k vertices.
  - ($m=|E|$). Any vertex covers at most k edges.

Kernelisation for Vertex Cover

H = G; (S = $\emptyset$;)

While there is a vertex v in H of degree at least k+1 do

Remove v and its incident edges from H

$k = k - 1$; (S = S + v ;)

If $k < 0$ then return **false**

If H has at least k^2+1 edges, then return **false**

Remove vertices of degree 0

Solve vertex cover on (H,k) with some algorithm

Time

- Kernelisation step can be done in O(n+m) time
- After kernelisation, we must solve the problem on a graph with at most k^2 edges, e.g., with branching this gives:
  - O($n + m + 2^k k^2$) time
  - O($kn + 2^k k^2$) time can be obtained by noting that there is no solution when $m > kn$.

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Better kernel for vertex cover

- Nemhauser-Trotter: kernel of at most *2k* vertices
- Make ILP formulation of Vertex Cover
- Solve relaxation
- All vertices v with $x_v > ½$: put v in set
- All vertices v with $x_v < ½$: v is not in the set
- Remove all vertices except those with value ½, and decrease k accordingly
- Gives kernel with at most *2k* vertices, but why is it correct?

Nemhauser Trotter proof plan

1. Write down the ILP for Vertex Cover
2. There is always an optimal solution of the relaxation with only values 1, 0 and ½
3. There is always an optimal solution of the ILP where all vertices with value 1 are in the vertex cover set and all vertices with value 0 are not in the vertex cover set
   - Compare the solution of part 2 with a hypothetical optimal solution of the ILP

ILP

$$\min \sum_{v \in V} x_v$$

$$\forall \{v, w\} \in E : x_v + x_w \geq 1$$

$$x_v \in \{0,1\}$$

Relaxation

$$\min \sum_{v \in V} x_v$$

$$\forall \{v, w\} \in E : x_v + x_w \geq 1$$

$$x_v \geq 0$$

There is always …

- … an optimal solution of the relaxation with only values 0, ½ and 1
- While not, repeat: take a vertex *v* with the largest value < 1, say c> ½. Look at the graph induced by vertices with weights c and 1-c.
- If the number of vertices with weights c and 1-c are not equal, the solution is not optimal.
- If these numbers are equal, change c to 1 and 1-c to 0.
- Repeat till we have the desired form

Vertices with weight 0 and 1

- There is an optimal solution of the ILP with vertices with weight 0 and weight 1 in the relaxation not changed
- A = weight in relaxation 1
- B = weight in relaxation 0
- C = weight in relaxation ½
- Note: no edges from B to C
- Take ILP solution x and relation y
  - Follow x on C and y on A and B: this is a solution
  - It is optimal, otherwise, taking y on C and x on A and B was not an optimal solution for the relaxation

2k kernel for Vertex Cover

- Solve the relaxation (polynomial time with the ellipsoid method, practical with Simplex)
- If the relaxation has optimum more than 2k, then say no
- Otherwise, get rid of the 0's and 1's, decrease k accordingly
- At most 2k vertices have weight ½ in the relaxation
- So, kernel has 2k vertices.
- It can (and will often) have a quadratic number of edges

Maximum Satisfiability

Given: Boolean formula in conjunctive normal form; integer k

Parameter: k

Question: Is there a truth assignment that satisfies at least k clauses?

- Denote: number of clauses: C

Reducing the number of clauses

- If $C \geq 2k$, then answer is YES
  - Look at arbitrary truth assignment, and truth assignment where we flip each value
  - Each clause is satisfied in one of these two assignment
  - So, one assignment satisfies at least half of all clauses

Bounding number of long clauses

- Long clause: has at least k literals
- Short clause: has at most k-1 literals
- Let L be number of long clauses
- If L $\geq k$: answer is YES
  - Select in each long clause a literal, whose complement is not yet selected
  - Set these all to true
  - All long clauses are satisfied

Reducing to only short clauses

- If less than k long clauses
  - Make new instance, with only the short clauses and k set to k-L
  - There is a truth assignment that satisfies at least k-L short clauses, if and only if there is a truth assignment that satisfies at least k clauses
    - =>: choose for each satisfied short clause a variable that makes the clause true. We may change all other variables, and can choose for each long clause another variable that makes it true
    - <=: trivial

An O(k^2) kernel for Maximum Satisfiability

- **If** at least $2k$ clauses **then** return YES
- **If** at least k long clauses **then** return YES
- **Else**
  - remove all L long clauses
  - set k=k-L

Kernelisation for cluster editing

- General form:
- Repeat rules, until no rule is possible
  - Rules can do some necessary modification and decrease k by one
  - Rules can remove some part of the graph
  - Rules can output YES or NO

Trivial rules and plan

- **Rule 1**: If a connected component of G is a clique, remove this connected component
- **Rule 2**: If we have more than k connected components and Rule 1 does not apply: Answer NO
- *Consequence*: after Rule 1 and Rule 2, there are at most k connected component
- *Plan*: find rules that make connected component small
- We change the input: some pairs are **permanent** and others are **forbidden**.

Observation and rule 3

- If two vertices *v*, *w* have *k*+1 neighbors in common, they must belong to the same clique in a solution
  - If the edge did not exist, add it and decrease *k* by 1
  - Set the edge {*v*,*w*} to be **permanent**

Another observation and rule 4

- If there are at least k+1 vertices that are adjacent to exactly one of v and w, then $\{v,w\}$ cannot be an edge in the solution
  - If $\{v,w\}$ is an edge: delete it and decrease k by one
  - Mark the pair $\{v,w\}$ as **forbidden**
- Rule 5: if a pair is **forbidden** and **permanent** then there is no solution

Transitivity

- Rule 6: if $\{v,w\}$ is permanent, and $\{w,x\}$ is permanent, then set $\{w,x\}$ to be permanent (if the edge was nonexisting, add it, and decrease k by one)
- Rule 7: if $\{v,w\}$ is permanent and $\{w,x\}$ is forbidden, then set $\{w,x\}$ to be forbidden (if the edge existed, delete it, and decrease k by one)

Counting

- Rules can be executed in polynomial time
- One can find in $O(n^3)$ time an instance to which no rules apply (with properly chosen data structures)
- Consider a connected component C with at least $4k+1$ vertices.
- At least $2k+1$ vertices are not involved in a modification, say this is the set W
- W must form a clique, and all edges in W become permanent

Counting continued

- Each vertex in C-W that is incident to k+1 or more vertices in W has a permanent edge to a vertex in W, and then gets permanent edges to all vertices in W, and then becomes member of W
- Each vertex in C-W for which at least k+1 vertices in W are not adjacent: it gets a forbidden edge to each vertex in W
- Each vertex in C-W is handled as $|W|>2k$.
- So, each connected component has size at most $4k$
- In total at most $4k^2$ vertices

Comments

- This argument is due to Gramm et al.
- Better and more recent algorithms exist: faster branching (2.7^k) and linear kernels

Non-blocker

- Given: graph G=(V,E), integer k
- Parameter k
- Question: Does G have a dominating set of size at most $|V|-k$?

Nonblocker kernels

- First idea: quadratic kernel
- Rules:
  1. If v has degree at least k, say YES
     - v and all vertices not a neighbor of v are a solution
  2. If v has degree 0, remove v
  3. If rules 1 and 2 do not apply, and we have more than $k(k+1)$ vertices, say YES
     - What would be the correct value here?

Lemma and simple kernel

- If G does not have vertices of degree 0, then G has a dominating set with at most |V|/2 vertices
  - Proof: per connected component: build spanning tree. The vertices on the odd levels form a ds, and the vertices on the even levels form a ds. Take the smaller of these.
- 2k kernel for non-blocker after removing vertices of degree 0

Improvements

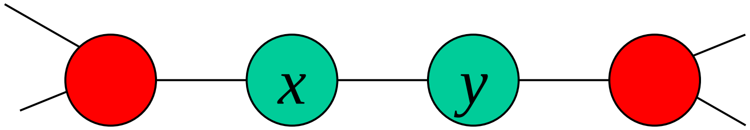
- Lemma (Blank and McCuaig, 1973) If a connected graph has minimum degree at least two and at least 8 vertices, then the size of a minimum dominating set is at most 2|V|/5.
- Lemma (Reed) If a connected graph has minimum degree at least three, then the size of a minimum dominating set is at most 3|V|/8.

Getting rid of vertices of degree 1

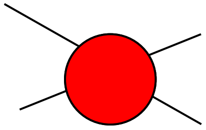
- Idea: we get rid of *all but one* vertices of degree 1
- Rule: if v and w have degree 1, then identify the neighbor of v and w and remove w
- Intuition: we can assume that a neighbor of a vertex of degree 1 is in the dominating set
- Possibly, replace in the end vertex of degree 1 by triangle

Degree two

- If we have an induced path like this:



- Then remove the two middle vertices and identify the endpoints



Iterative compression

- FPT-technique

Feedback Vertex Set

- Instance: graph G=(V,E)
- Parameter: integer k
- Question: Is there a set of at most k vertices W, such that G-W is a forest?
  - Known in FPT
  - Here: recent algorithm $O(5^k\, p(n))$ time algorithm
  - Can be done in $O(5^k\, kn)$ or less with kernelisation

Iterative compression technique

- Number vertices $v_1, v_2, \ldots, v_n$
- Let X = $\{v_1, v_2, \ldots, v_k\}$
- **for** $i = k+1$ **to** n **do**
  - Add v_i to X
    - Note: X is a FVS of size at most $k+1$ of $\{v_1, v_2, \ldots, v_i\}$
  - Call a subroutine that either
    - Finds (with help of X) a feedback vertex set Y of size at most k in $\{v_1, v_2, \ldots, v_i\}$; set X = Y OR
    - Determines that Y does not exist; stop, return NO

Compression subroutine

- Given: graph G, FVS X of size $k + 1$
- Question: find if existing FVS of size k
  - – Is subroutine of main algorithm

➢ **for** all subsets S of X **do**

Determine if there is a FVS of size at most k that contains all vertices in S and no vertex in X – S

Yet a deeper subroutine

- Given: Graph G, FVS X of size k+1, set S
- Question: find if existing a FVS of size k containing all vertices in S and no vertex from X – S

1. Remove all vertices in S from G
2. Mark all vertices in X – S
3. If marked cycles contain a cycle, then return NO
4. While marked vertices are adjacent, contract them
5. Set $k = k - |S|$. If $k < 0$, then return NO
6. If G is a forest, then return YES; S
7. …

Subroutine continued

7. If an unmarked vertex v has at least two edges to marked vertices
   - If these edges are parallel, i.e., to the same neighbor, then v must be in a FVS (we have a cycle with v the only unmarked vertex)
     - Put v in S, set $k = k - 1$ and recurse
   - Else recurse twice:
     - Put v in S, set $k = k - 1$ and recurse
     - Mark v, contract v with all marked neighbors and recurse
       - The number of marked vertices is one smaller

Other case

8. Choose an unmarked vertex *v* that has at most one unmarked neighbor (a leaf in G[V-X])
    - ➢ By step 7, it also has at most one marked neighbor

- If *v* is a leaf in G, then remove *v*
- If *v* has degree 2, then remove *v* and connect its neighbors

Analysis

- Precise analysis gives $O^*(5^k)$ subproblems in total
- Imprecise: 2^k subsets S
- Only branching step:
  - k is decreased by one, or
  - Number of marked vertices is decreased by one
- Initially: number of marked vertices + k is at most $2k$
- Bounded by $2^k.2^{2k} = 8^k$

Conclusions

- Similar techniques work (usually much more complicated) for many other problems
- W[…]-hardness results indicate that FPT-algorithms do not exist for other problems
- Note similarities and differences with exponential time algorithms