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Usage Analysis

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Type and effect systems - Introduction

Type-based approaches to static program analysis

- Static program analysis: compile-time techniques for approximating the set of values or behaviours that arise at run-time when a program is executed.
- Applications: verification, optimization.
- Different approaches: data-flow analysis, constraint-based analysis, abstract interpretation, type-based analysis.
- Type-based analysis: equipping a programming language with a nonstandard type system that keeps track of some properties of interest.
- Advantages: reuse of tools, techniques, and infrastructure (polymorphism, subtyping, type inference, . . .).
- Focus: accuracy vs. modularity.

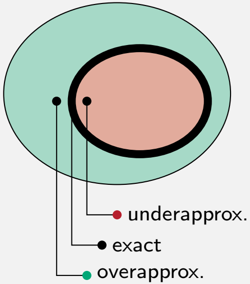
Examples

. . .
Side-effect analysis.
Callability analysis.
Reachability analysis.
Uniqueness analysis.
Totality analysis.
Security analysis.
Strictness analysis.
Sharing analysis.
Trust analysis.
. . .

. . .
Sign analysis.
Flow analysis.
Control-flow analysis.
Class-hierarchy analysis.
Region analysis.
Binding-time analysis.
Alias analysis.
Communication analysis .
Escape analysis.
. . .

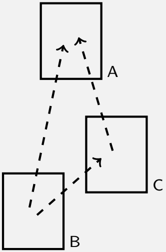
Accuracy

- Establishing nontrivial properties of programs is in general undecidable (halting problem, Rice's theorem).
- In static analysis we have to settle for "useful" approximations of properties.
- "Useful" means: sound ("erring at the safe side") and accurate (as precise as possible).



Modularity

- Breaking up a (large) program in smaller units or modules is generally considered good programming style.
- Separate compilation: compile each module in isolation.
- Advantage: only modules that have been edited need to be recompiled.
- To facilitate seperate compilation, each unit of compilation needs to be analysed in isolation, i.e., without knowledge of how it's used from within the rest of the program.



☞ Tension between accuracy and modularity: whole-program analysis typically yields more precise results.

1. Introduction to usage analysis

Usage analysis

- Usage analysis: determining which objects in a (functional) program are guaranteed to be used at most once and—dually— which objects may be used more than once.
- Two flavours: uniqueness analysis (a.k.a. uniquenss typing) and sharing analysis.
- Hage et al. (ICFP 2007): A generic usage analysis with subeffect qualifiers.

Hage et al. (ICFP 2007): abstract

"Sharing analysis and uniqueness typing are static analyses that aim at determining which of a program's objects are to be used at most once. There are many commonalities between these two forms of usage analysis. We make their connection precise by developing an expressive generic analysis that can be instantiated to both sharing analysis and uniqueness typing. The resulting system, which combines parametric polymorphism with effect subsumption, is specifed within the general framework of qualified types, so that readily available tools and techniques can be used for the development of implementations and metatheory."

Destructive updates

- An important property of pure functional languages is referential transparency: a given expression will yield one and the same value each time it is evaluated.
- Referential transparency enables equational reasoning.
- But some operations are destructive by nature: for example, altering the contents of a file.
- Such destructive operations break referential transparency.

Problems with destructive updates

Simple I/O interface:

$$\begin{aligned} \mathit{readFile} &:: \mathit{String} \to \mathit{File} \\ \mathit{fPutChar} &:: \mathit{Char} \to \mathit{File} \to \mathit{File} \end{aligned}$$

For example:

$$\begin{aligned} &\mathbf{let}\ f = \mathit{readFile}\ \texttt{"DATA"} \\ &\mathbf{in}\ \ (\mathit{fPutChar}\ \texttt{'O'}\ f, \mathit{fPutChar}\ \texttt{'K'}\ f) \end{aligned}$$

☞ What is the meaning of this program? (Assume lazy evaluation.)

"Safe" destructive updates

Idea: referential transparency can be recovered if we restrict destructive updates to operations that hold the only reference to the object that is to be destructed.

Example:

$$\textbf{let}\ f = \mathit{readFile}\ \texttt{"DATA"}$$
$$\textbf{in}\ \ (\mathit{fPutChar}\ \texttt{'K'} \circ \mathit{fPutChar}\ \texttt{'O'})\ f$$

☞ Each file handle is used at most once.

Self-updating closures

- Lazy evaluation is typically implemented by means of self-updating closures.
- For example:

$$(\lambda x \to x + x)\ (2 + 3)$$

- A closure is created for the expression $(2 + 3)$ and associated with x.
- When x is first accessed, the closure evaluates its expression and updates itself with the result (5).
- For the second access of x, the closure can immediately produce the value 5.
- The update avoids re-evaluation of $(2 + 3)$.

Unnecessary updates

Another example:

$$(\lambda x \rightarrow 2 * x)\ (2 + 3)$$

☞ Now, the update of the closure is unneccesary, because x is accessed only once.

Two flavours of usage analysis

Uniqueness analysis:

- Determines which objects have at most one reference.
- Application: destructive updates that are "safe" w.r.t. referential transparency.
- Used in Clean as an alternative to monads.

Sharing analysis:

- Determines which function arguments are accessed at most once.
- Application: avoiding unneccesary closure updates.
- For other applications, see Turner et al. (FPCA 1995), Wansbrough and Peyton Jones (POPL 1999), and Gustavsson and Sands (ENTCS 26).

Generic usage analysis

- Both uniqueness analysis and sharing analysis aim at keeping track of objects that are used at most once.
- If we forget about modularity and settle for little accuracy, we can use a single nonstandard type system for both analyses.
- For more realistic requirements, we can still define a single parameterized type system that can be instantiated to uniqueness analysis as well as sharing analysis.

2. The underlying type system

Term language

- It would be impractical to define the analysis for a full-fledged language like Haskell or Clean.
- Instead, we use a small toy language.

n	$\in$	$\mathbf{Num}$	numerals
x	$\in$	$\mathbf{Var}$	variables
t	$\in$	$\mathbf{Tm}$	terms
v	$\in$	$\mathbf{Val} \subset \mathbf{Tm}$	values

$$\begin{array}{lcl} t & ::= & n \mid x \mid \lambda x.\, t_1 \mid t_1\ t_2 \mid \mathbf{let}\ x = t_1\ \mathbf{in}\ t_2\ \mathbf{ni} \\ & \mid & t_1 + t_2 \\ v & ::= & n \mid \lambda x.\, t_1 \end{array}$$

Natural semantics

- The meaning of programs is defined by means of a so-called big-step or natural semantics.
- Evaluation relation: judgements of the form $t \longrightarrow v$.
- Rules are given in natural deduction style:

$$\frac{\mathit{hyp}_1 \quad \cdots \quad \mathit{hyp}_n}{\mathit{concl}}$$

Natural semantics: numerals and abstractions

Numerals and abstractions are already values:

$$\frac{}{n \longrightarrow n}$$

$$\frac{}{\lambda x.\, t_1 \longrightarrow \lambda x.\, t_1}$$

Natural semantics: applications

Beta-reduction:

$$\frac{t_1 \longrightarrow \lambda x.\, t_{11} \qquad [x \mapsto t_2]t_{11} \longrightarrow v}{t_1\ t_2 \longrightarrow v}$$

☞ $[x \mapsto t_2]t_{11}$ means
"replace each free occurrence of x in t_{11} by t_2".

☞ Lazy evaluation: arguments are passed unevaluated.

Natural semantics: local definitions

Local definitions are also evaluated by means of beta-reduction:

$$\frac{[x \mapsto t_1]\,t_2 \longrightarrow v}{\textbf{let } x = t_1 \textbf{ in } t_2 \textbf{ ni} \longrightarrow v}\ [\textit{e-let}]$$

☞ Local definitions are evaluated as if

$$\textbf{let } x = t_1 \textbf{ in } t_2 \qquad \equiv \qquad (\lambda x.\, t_2)\ t_1.$$

Natural semantics: addition

Addition is strict, i.e., it first evaluated both its operands:

$$\frac{t_1 \longrightarrow n_1 \qquad t_2 \longrightarrow n_2 \qquad n_1 \oplus n_2 = n}{t_1 + t_2 \longrightarrow n}$$

☞ $\oplus$ denotes "ordinary" addition of natural numbers.

Types and type environments

- Types are built from the type Nat of natural numbers and the function-type constructor $\rightarrow$.
- Type environments map variables to types.

$$
\begin{array}{lll}
\tau & \in & \mathbf{Ty} \quad \text{types} \\
\Gamma & \in & \mathbf{TyEnv} \quad \text{type environments}
\end{array}
$$

$$
\begin{array}{lll}
\tau & ::= & Nat \mid \tau_1 \rightarrow \tau_2 \\
\Gamma & ::= & [\,] \mid \Gamma_1[x \mapsto \tau]
\end{array}
$$

- We write $\Gamma(x) = \tau$ if the rightmost binding for x in Γ associates to τ.

Typing

- We approximate the set of "well-behaved" programs by means of a type system.
- Typing relation: judgements of the form $\Gamma \vdash_{\mathsf{UL}} t : \tau$.
- "In type environment Γ, the term t can be assigned the type τ."
- Γ is supposed to contain types for the free variables of t.
- The subscript UL is used to distinguish the judgements of this underlying type system from the (nonstandard) type systems we will consider later on.

Typing: numerals and variables

- Each numeral denotes a natural number:

$$\frac{}{\Gamma \vdash_{\mathsf{UL}} n : Nat}$$

- The type of a variable should be made available through the type environment Γ:

$$\frac{\Gamma(x) = \tau}{\Gamma \vdash_{\mathsf{UL}} x : \tau}$$

Typing: abstractions and applications

- To type a lambda-abstraction, we "guess" a type for its parameter and extend the type environment accordingly:

$$\frac{\Gamma[x \mapsto \tau_1] \vdash_{\mathsf{UL}} t_1 : \tau_2}{\Gamma \vdash_{\mathsf{UL}} \lambda x.\, t_1 : \tau_1 \to \tau_2}$$

- For a function application to be well-typed, the type of the argument needs to match the domain of the function.
- The type of the application is then determined by the codomain of the function.

$$\frac{\Gamma \vdash_{\mathsf{UL}} t_1 : \tau_2 \to \tau \qquad \Gamma \vdash_{\mathsf{UL}} t_2 : \tau_2}{\Gamma \vdash_{\mathsf{UL}} t_1\ t_2 : \tau}$$

Typing: local definitions

To type the body of a local definition, we extend the type environment with the type of the defined value:

$$\frac{\Gamma \vdash_{\mathsf{UL}} t_1 : \tau_1 \quad \Gamma[x \mapsto \tau_1] \vdash_{\mathsf{UL}} t_2 : \tau}{\Gamma \vdash_{\mathsf{UL}} \textbf{let}\ x = t_1\ \textbf{in}\ t_2\ \textbf{ni} : \tau}$$

☞ Local definitions are typed as if

$$\textbf{let}\ x = t_1\ \textbf{in}\ t_2 \qquad \equiv \qquad (\lambda x.\, t_2)\ t_1.$$

Typing: addition

Addition requires that both operands are natural numbers. The result is a natural number as well.

$$\frac{\Gamma \vdash_{\mathsf{UL}} t_1 : \mathit{Nat} \quad \Gamma \vdash_{\mathsf{UL}} t_2 : \mathit{Nat}}{\Gamma \vdash_{\mathsf{UL}} t_1 + t_2 : \mathit{Nat}}$$

3. Polymorphism

Evaluation and typing

- The evaluation and typing rules constitute logics.
- The rules are used to construct proofs of judgements of the forms $t \longrightarrow v$ and $\Gamma \vdash_{\mathsf{UL}} t : \tau$.

Evaluation: examples

$$2 \longrightarrow 2$$

$$2 + 1 \longrightarrow 3$$

$$[x \mapsto 2](x + 1) \longrightarrow 3$$

$$\lambda x.\, x + 1 \longrightarrow \lambda x.\, x + 1$$

$$(\lambda x.\, x + 1)\ 2 \longrightarrow 3$$

Typing: examples

$$[\,] \vdash_{\mathsf{UL}} 2 : \mathit{Nat}$$

$$[\,] \vdash_{\mathsf{UL}} 2 + 1 : \mathit{Nat}$$

$$[x \mapsto \mathit{Nat}] \vdash_{\mathsf{UL}} x + 1 : \mathit{Nat}$$

$$[\,] \vdash_{\mathsf{UL}} \lambda x.\, x + 1 : \mathit{Nat} \to \mathit{Nat}$$

$$[\,] \vdash_{\mathsf{UL}} (\lambda x.\, x + 1)\ 2 : \mathit{Nat}$$

Evaluation: proof tree

$$\dfrac{\dfrac{}{\lambda x.\,x+1 \longrightarrow \lambda x.\,x+1} \qquad \dfrac{\dfrac{}{2 \longrightarrow 2} \quad \dfrac{}{1 \longrightarrow 1} \quad 2 \oplus 1 = 3}{[x \mapsto 2](x+1) \longrightarrow 3}}{(\lambda x.\,x+1)\ 2 \longrightarrow 3}$$

Typing: proof tree

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{[x \mapsto Nat](x) = Nat}{[x \mapsto Nat] \vdash_{\mathsf{UL}} x : Nat}
      \qquad
      \dfrac{}{[x \mapsto Nat] \vdash_{\mathsf{UL}} 1 : Nat}
    }{[x \mapsto Nat] \vdash_{\mathsf{UL}} x + 1 : Nat}
  }{[\,] \vdash_{\mathsf{UL}} \lambda x.\, x + 1 : Nat \to Nat}
  \qquad
  \dfrac{}{[\,] \vdash_{\mathsf{UL}} 2 : Nat}
}{[\,] \vdash_{\mathsf{UL}} (\lambda x.\, x + 1)\ 2 : Nat}
$$

Some terms have multiple types

$$\vdots$$

$$\dfrac{\dfrac{[x \mapsto \mathit{Nat}](x) = \mathit{Nat}}{[x \mapsto \mathit{Nat}] \vdash_{\mathsf{UL}} x : \mathit{Nat}}}{[\,] \vdash_{\mathsf{UL}} \lambda x.\, x : \mathit{Nat} \to \mathit{Nat}}$$

$$\vdots$$

$$\dfrac{\dfrac{[x \mapsto \mathit{Nat} \to \mathit{Nat}](x) = \mathit{Nat} \to \mathit{Nat}}{[x \mapsto \mathit{Nat} \to \mathit{Nat}] \vdash_{\mathsf{UL}} x : \mathit{Nat} \to \mathit{Nat}}}{[\,] \vdash_{\mathsf{UL}} \lambda x.\, x : (\mathit{Nat} \to \mathit{Nat}) \to \mathit{Nat} \to \mathit{Nat}}$$

$$\vdots$$

Multiple types and modularity

- Which type makes sense for $id = \lambda x.\, x$ depends on the context in which it is used.
  - For $id\ 2$ it is $Nat \to Nat$.
  - For $id\ (\lambda x.\, x + 1)\ 2$ it is $(Nat \to Nat) \to Nat \to Nat$.
  - What about $id\ 2 + id\ (\lambda x.\, x + 1)\ 2$?
- To type id independent from its context, we assign it a polymorphic type: $\forall \alpha.\, \alpha \to \alpha$.
- Polymorphic types are obtained by generalizing from concrete types.
- Polymorphic types can be instantiated to concrete, monomorphic types.
- Instantiating α to Nat in $\forall \alpha.\, \alpha \to \alpha$ yields $Nat \to Nat$.
- Instantiating α to $Nat \to Nat$ in $\forall \alpha.\, \alpha \to \alpha$ yields $(Nat \to Nat) \to Nat \to Nat$.

The Damas-Milner discipline

- Let-polymorphism à la Damas and Milner is on a sweet spot in the design space of typed languages.
- It allows for modularity by assigning polymorphic types to terms.
- Still, all types can be inferred, i.e., derived by the compiler without any help from the programmer.
  - Peyton Jones et al: the sweet spot may have turned sour.
- If a term is typeable, it has a principal type: a most general (i.e., most polymorphic) type.
- All assignable concrete types can be obtained from the principal type by instantiation.

Polymorphic typing: syntax

$$
\begin{array}{lll}
\alpha & \in & \mathbf{TyVar} & \text{type variables} \\
\sigma & \in & \mathbf{TyScheme} & \text{type schemes}
\end{array}
$$

$$
\begin{array}{lll}
\tau & ::= & \alpha \mid Nat \mid \tau_1 \to \tau_2 \\
\sigma & ::= & \tau \mid \forall \alpha.\, \sigma_1 \\
\Gamma & ::= & [\,] \mid \Gamma_1[x \mapsto \sigma]
\end{array}
$$

Polymorphic typing: typing relation

- A Damas-Milner style polymorphic type system can be obtained from our previous, monomorphic type system by adapting the rules for variables and local definitions adding rules for generalization and instantiation.
- The judgements of the typing relation are now of the form $\Gamma \vdash_{\mathsf{DM}} t : \sigma$.

Polymorphic typing: variables and definitions

Type environments map to type schemes now:

$$\frac{\Gamma(x) = \sigma}{\Gamma \vdash_{\mathsf{DM}} x : \sigma}$$

Definitions can be assigned polymorphic types:

$$\frac{\Gamma \vdash_{\mathsf{DM}} t_1 : \sigma_1 \quad \Gamma[x \mapsto \sigma_1] \vdash_{\mathsf{DM}} t_2 : \tau}{\Gamma \vdash_{\mathsf{DM}} \mathbf{let}\ x = t_1\ \mathbf{in}\ t_2\ \mathbf{ni} : \tau}$$

☞ In all other rules, just replace $\vdash_{\mathsf{UL}}$ by $\vdash_{\mathsf{DM}}$.

Polymorphic typing: new rules

Generalization:

$$\frac{\Gamma \vdash_{\mathsf{DM}} t : \sigma_1 \quad \alpha \notin \mathit{ftv}(\Gamma)}{\Gamma \vdash_{\mathsf{DM}} t : \forall \alpha.\, \sigma_1}$$

Instantiation:

$$\frac{\Gamma \vdash_{\mathsf{DM}} t : \forall \alpha.\, \sigma_1}{\Gamma \vdash_{\mathsf{DM}} t : [\alpha \mapsto \tau_0]\sigma_1}$$

☞ $\mathit{ftv}(\Gamma)$ retrieves all free variables from Γ.

Generalization: example

$$\dfrac{\dfrac{\dfrac{[x \mapsto \alpha](x) = \alpha}{[x \mapsto \alpha] \vdash_{\mathsf{DM}} x : \alpha}}{[\,] \vdash_{\mathsf{DM}} \lambda x.\, x : \alpha \to \alpha} \qquad \alpha \notin \mathit{ftv}([\,])}{[\,] \vdash_{\mathsf{DM}} \lambda x.\, x : \forall \alpha.\, \alpha \to \alpha}$$

Instantiation: example

Let $\Gamma = [id \mapsto \forall\alpha.\,\alpha \to \alpha]$:

$$\dfrac{\dfrac{\dfrac{\Gamma(id) = \forall\alpha.\,\alpha \to \alpha}{\Gamma \vdash_{\mathsf{DM}} id : \forall\alpha.\,\alpha \to \alpha}}{\Gamma \vdash_{\mathsf{DM}} id : Nat \to Nat} \qquad \dfrac{}{\Gamma \vdash_{\mathsf{DM}} 2 : Nat}}{\Gamma \vdash_{\mathsf{DM}} id\ 2 : Nat}$$

Exercise

Give a proof tree for

$$[\,] \vdash_{\mathsf{DM}} \mathbf{let}\ id = \lambda x.\, x\ \mathbf{in}\ id\ 2 + id\ (\lambda x.\, x + 1)\ 2\ \mathbf{ni} : Nat$$

4. The analysis

Examples

$$(\lambda x.\, x + 1)\; 2$$

2 is used at most once.

$$(\lambda x.\, x + x)\; 2$$

2 is used more than once.

$$(\lambda x.\, \lambda y.\, x)\; 2\; 3$$

2 is used at most once; 3 is used at most once.

$$(\lambda f.\, \lambda x.\, f\;\; x)\; (\lambda y.\, y + y)\; 2$$

2 is used more than once.

Annotated type system

- Our usage analysis will be specified as an annotated type system.
- We extend the Damas-Milner type system by annotating types, type environments, and typing judgements with information on how often a term is used.
- Two annotations: 1 and ω.
- 1: the term is guaranteed to be used at most once.
- ω: the term may be used more than once.
- Judgements have the form $\widehat{\Gamma} \vdash_{\mathsf{UA}} t :^{\varphi} \widehat{\sigma}$.
- φ ranges over annotations.
- $\widehat{\Gamma}$ ranges over annotated type environments.
- $\widehat{\sigma}$ ranges over annotated type schemes.

Usage analysis: syntax

φ	$\in$	$\mathbf{Ann}$	annotations
$\widehat{\tau}$	$\in$	$\widehat{\mathbf{Ty}}$	annotated types
$\widehat{\sigma}$	$\in$	$\widehat{\mathbf{TyScheme}}$	annotated type schemes
$\widehat{\Gamma}$	$\in$	$\widehat{\mathbf{TyEnv}}$	annotated type environments

$$
\begin{array}{lcl}
\varphi & ::= & 1 \mid \omega \\
\widehat{\tau} & ::= & \alpha \mid \mathit{Nat} \mid \widehat{\tau}_1{}^{\varphi_1} \to \widehat{\tau}_2{}^{\varphi_2} \\
\widehat{\sigma} & ::= & \widehat{\tau} \mid \forall \alpha.\, \widehat{\sigma}_1 \\
\widehat{\Gamma} & ::= & [\,] \mid \widehat{\Gamma}_1[x \mapsto^{\varphi} \widehat{\sigma}]
\end{array}
$$

- We write $\widehat{\Gamma}(x) =^{\varphi} \widehat{\sigma}$ if the rightmost binding for x in $\widehat{\Gamma}$ associates to φ and $\widehat{\sigma}$.
- We write $\widehat{\Gamma} \setminus x$ for the environment obtained by removing all bindings for x from $\widehat{\Gamma}$.

Usage analysis: numerals

It depends on the context of a numeral whether it used at most once:

$$\frac{}{\widehat{\Gamma} \vdash_{\mathsf{UA}} n :^{1} \mathit{Nat}}$$

—or possibly more than once:

$$\frac{}{\widehat{\Gamma} \vdash_{\mathsf{UA}} n :^{\omega} \mathit{Nat}}$$

Merging the two rules:

$$\frac{}{\widehat{\Gamma} \vdash_{\mathsf{UA}} n :^{\varphi} \mathit{Nat}}$$

Usage analysis: variables

To analyse a variable, we look it up in the environment:

$$\frac{\widehat{\Gamma}(x) =^{\varphi} \widehat{\sigma}}{\widehat{\Gamma} \vdash_{\mathsf{UA}} x :^{\varphi} \widehat{\sigma}}$$

The rôle of environments

An annotated type environment should reflect how often the free variables of a term are used:

$$[x \mapsto^{1} Nat] \vdash_{\mathsf{UA}} x + 1 :^{\varphi} Nat$$

should be valid.

$$[x \mapsto^{1} Nat] \vdash_{\mathsf{UA}} x + x :^{\varphi} Nat$$

should not be valid.

$$[x \mapsto^{\omega} Nat] \vdash_{\mathsf{UA}} x + 1 :^{\varphi} Nat$$

should be valid.

$$[x \mapsto^{\omega} Nat] \vdash_{\mathsf{UA}} x + x :^{\varphi} Nat$$

should be valid.

Context splitting

- **Idea:** for every possible branch in a term's control-flow graph (for example a function application or an addition), we split the type environment in a left and a right part: $\widehat{\Gamma} \sim_{\mathsf{UA}} \widehat{\Gamma}_1 \bowtie \widehat{\Gamma}_2$.
- Bindings for 1-annotated variables go either left or right.
- Bindings for ω-annotated variables may go both ways.

Context splitting: rules

$$\frac{}{[\,] \sim_{\mathsf{UA}} [\,] \bowtie [\,]}$$

$$\frac{\widehat{\Gamma}_1 \sim_{\mathsf{UA}} \widehat{\Gamma}_{11} \bowtie \widehat{\Gamma}_{12}}{\widehat{\Gamma}_1[x \mapsto^{\varphi} \widehat{\sigma}] \sim_{\mathsf{UA}} \widehat{\Gamma}_{11}[x \mapsto^{\varphi} \widehat{\sigma}] \bowtie \widehat{\Gamma}_{12} \setminus x}$$

$$\frac{\widehat{\Gamma}_1 \sim_{\mathsf{UA}} \widehat{\Gamma}_{11} \bowtie \widehat{\Gamma}_{12}}{\widehat{\Gamma}_1[x \mapsto^{\varphi} \widehat{\sigma}] \sim_{\mathsf{UA}} \widehat{\Gamma}_{11} \setminus x \bowtie \widehat{\Gamma}_{12}[x \mapsto^{\varphi} \widehat{\sigma}]}$$

$$\frac{\widehat{\Gamma}_1 \sim_{\mathsf{UA}} \widehat{\Gamma}_{11} \bowtie \widehat{\Gamma}_{12}}{\widehat{\Gamma}_1[x \mapsto^{\omega} \widehat{\sigma}] \sim_{\mathsf{UA}} \widehat{\Gamma}_{11}[x \mapsto^{\omega} \widehat{\sigma}] \bowtie \widehat{\Gamma}_{12}[x \mapsto^{\omega} \widehat{\sigma}]}$$

Usage analysis: addition

$$\frac{\widehat{\Gamma} \sim_{\mathsf{UA}} \widehat{\Gamma}_1 \bowtie \widehat{\Gamma}_2 \quad \widehat{\Gamma}_1 \vdash_{\mathsf{UA}} t_1 :^{\varphi_1} \mathit{Nat} \quad \widehat{\Gamma}_2 \vdash_{\mathsf{UA}} t_2 :^{\varphi_2} \mathit{Nat}}{\widehat{\Gamma} \vdash_{\mathsf{UA}} t_1 + t_2 :^{\varphi} \mathit{Nat}}$$

☞ If a variable is used in both t_1 and t_2, context splitting guarantees that it is ω-annotated in $\widehat{\Gamma}$.

Usage analysis: example

$$\dfrac{\dfrac{\dfrac{\widehat{\Gamma}_{11}(x) =^{\omega} \mathit{Nat}}{\widehat{\Gamma}_{11} \vdash_{\mathsf{UA}} x :^{\omega} \mathit{Nat}} \qquad \dfrac{\widehat{\Gamma}_{12}(y) =^{1} \mathit{Nat}}{\widehat{\Gamma}_{12} \vdash_{\mathsf{UA}} y :^{1} \mathit{Nat}}}{\widehat{\Gamma}_{1} \vdash_{\mathsf{UA}} x + y :^{1} \mathit{Nat}} \qquad \dfrac{\widehat{\Gamma}_{2}(x) =^{\omega} \mathit{Nat}}{\widehat{\Gamma}_{2} \vdash_{\mathsf{UA}} x :^{\omega} \mathit{Nat}}}{[x \mapsto^{\omega} \mathit{Nat}, y \mapsto^{1} \mathit{Nat}, z \mapsto^{1} \mathit{Nat}] \vdash_{\mathsf{UA}} (x + y) + x :^{1} \mathit{Nat}}$$

(context splits omitted)

$$\begin{array}{ll}
\widehat{\Gamma}_{1} & = [x \mapsto^{\omega} \mathit{Nat}, y \mapsto^{1} \mathit{Nat}, z \mapsto^{1} \mathit{Nat}] \\
\widehat{\Gamma}_{11} & = [x \mapsto^{\omega} \mathit{Nat}, \phantom{y \mapsto^{1} \mathit{Nat},} z \mapsto^{1} \mathit{Nat}] \\
\widehat{\Gamma}_{12} & = [x \mapsto^{\omega} \mathit{Nat}, y \mapsto^{1} \mathit{Nat} \phantom{, z \mapsto^{1} \mathit{Nat}}] \\
 & \\
\widehat{\Gamma}_{2} & = [x \mapsto^{\omega} \mathit{Nat} \phantom{, y \mapsto^{1} \mathit{Nat}, z \mapsto^{1} \mathit{Nat}}]
\end{array}$$

Usage analysis: local definitions

$$\frac{\widehat{\Gamma} \sim_{\mathsf{UA}} \widehat{\Gamma}_1 \bowtie \widehat{\Gamma}_2 \quad \widehat{\Gamma}_1 \vdash_{\mathsf{UA}} t_1 :^{\varphi_1} \widehat{\sigma}_1 \quad \widehat{\Gamma}_2[x \mapsto^{\varphi_1} \widehat{\sigma}_1] \vdash_{\mathsf{UA}} t_2 :^{\varphi} \widehat{\tau}}{\widehat{\Gamma} \vdash_{\mathsf{UA}} \mathbf{let}\ x = t_1\ \mathbf{in}\ t_2\ \mathbf{ni} :^{\varphi} \widehat{\tau}}$$

Usage analysis: applications

$$\frac{\widehat{\Gamma} \sim_{\mathsf{UA}} \widehat{\Gamma}_1 \bowtie \widehat{\Gamma}_2 \quad \widehat{\Gamma}_1 \vdash_{\mathsf{UA}} t_1 :^{\varphi_1} \widehat{\tau}_2{}^{\varphi_2} \rightarrow \widehat{\tau}^{\varphi} \quad \widehat{\Gamma}_2 \vdash_{\mathsf{UA}} t_2 :^{\varphi_2} \widehat{\tau}_2}{\widehat{\Gamma} \vdash_{\mathsf{UA}} t_1\ t_2 :^{\varphi} \widehat{\tau}}$$

- Domain and domain annotation should match type and usage of argument.
- Result type and usage of application are retrieved from codomain and codomain annotation.

Usage analysis: abstractions (first attempt)

$$\frac{\widehat{\Gamma}[x \mapsto^{\varphi_1} \widehat{\tau}_1] \vdash_{\mathsf{UA}} t_1 :^{\varphi_2} \widehat{\tau}_2}{\widehat{\Gamma} \vdash_{\mathsf{UA}} \lambda x.\, t_1 :^{\varphi} \widehat{\tau}_1{}^{\varphi_1} \to \widehat{\tau}_2{}^{\varphi_2}}$$

For example:

$$[\,] \vdash_{\mathsf{UA}} \lambda x.\, x + 1 :^{1} \mathit{Nat}^{1} \to \mathit{Nat}^{1}$$

$$[\,] \vdash_{\mathsf{UA}} \lambda x.\, x + x :^{1} \mathit{Nat}^{\omega} \to \mathit{Nat}^{1}$$

Partial applications: problem

$$
\begin{array}{l}
\textbf{let } f = \lambda x.\,\lambda y.\, x + y \\
\textbf{in } \ \textbf{let } g = f\ (2+3) \\
\quad\ \ \textbf{in } \ g\ 7 + g\ 11 \\
\quad\ \ \textbf{ni} \\
\textbf{ni}
\end{array}
$$

- How often is g used?
- How often is $(2+3)$ used?
- $Nat^1 \rightarrow (Nat^1 \rightarrow Nat^1)^\omega$ is a valid type for f. Should it be?

Containment

- **Containment:** an object is potentially used as least as often as an object it is contained in.

$$
\begin{array}{l}
\textbf{let } f = \lambda x.\lambda y.\, x + y \\
\textbf{in } \ \textbf{let } g = f\ (2+3) \\
\quad\quad \textbf{in } \ g\ 7 + g\ 11 \\
\quad\quad \textbf{ni} \\
\textbf{ni}
\end{array}
$$

- The binding of x to $(2+3)$ is contained in the partial application g.
- The partial application is used more than once: hence, so is $(2+3)$.

Usage analysis: abstractions (another look)

$$\frac{\widehat{\Gamma}[x \mapsto^{\varphi_1} \widehat{\tau}_1] \vdash_{\mathsf{UA}} t_1 :^{\varphi_2} \widehat{\tau}_2}{\widehat{\Gamma} \vdash_{\mathsf{UA}} \lambda x.\, t_1 :^{\varphi} \widehat{\tau}_1{}^{\varphi_1} \to \widehat{\tau}_2{}^{\varphi_2}}$$

- **Problem:** the free variables of the abstraction could be used as least as often as the abstraction itself.
- The usage of the free variables is reflected by $\widehat{\Gamma}$.
- The usage of the abstraction is reflected by φ.
- **Solution:** If $\varphi \equiv \omega$, then all bindings in $\widehat{\Gamma}$ that are used in the typing of t_1 should also be ω.

Usage analysis: abstractions (refined)

$$\frac{\widehat{\Gamma} \triangleright^{\varphi} \widehat{\Gamma}_{11} \quad \widehat{\Gamma}_{11}[x \mapsto^{\varphi_1} \widehat{\tau}_1] \vdash_{\mathsf{UA}} t_1 :^{\varphi_2} \widehat{\tau}_2}{\widehat{\Gamma} \vdash_{\mathsf{UA}} \lambda x.\, t_1 :^{\varphi} \widehat{\tau}_1{}^{\varphi_1} \to \widehat{\tau}_2{}^{\varphi_2}}$$

$\widehat{\Gamma} \triangleright^{\varphi} \widehat{\Gamma}_{11}$:

- $\widehat{\Gamma}_{11}$ is a subenvironment of $\widehat{\Gamma}$;
- if $\varphi \equiv \omega$, then all bindings in $\widehat{\Gamma}_{11}$ are annotated with ω.

Containment: rules

$$\frac{}{[\,] \rhd^{\varphi} [\,]}$$

$$\frac{\widehat{\Gamma}_{11} \rhd^{\varphi} \widehat{\Gamma}_{2}}{\widehat{\Gamma}_{11}[x \mapsto^{\varphi_0} \widehat{\sigma}] \rhd^{\varphi} \widehat{\Gamma}_{2}}$$

$$\frac{\widehat{\Gamma}_{11} \rhd^{1} \widehat{\Gamma}_{2}}{\widehat{\Gamma}_{11}[x \mapsto^{\varphi_0} \widehat{\sigma}] \rhd^{1} \widehat{\Gamma}_{2}[x \mapsto^{\varphi_0} \widehat{\sigma}]}$$

$$\frac{\widehat{\Gamma}_{11} \rhd^{\omega} \widehat{\Gamma}_{2}}{\widehat{\Gamma}_{11}[x \mapsto^{\omega} \widehat{\sigma}] \rhd^{\omega} \widehat{\Gamma}_{2}[x \mapsto^{\omega} \widehat{\sigma}]}$$

Containment: examples

$$
\begin{array}{l}
\textbf{let } f = \lambda x.\, \lambda y.\, x + y \\
\textbf{in } \ \textbf{let } g = f\ (2+3) \\
\qquad \textbf{in } \ g\ 7 + g\ 11 \\
\qquad \textbf{ni} \\
\textbf{ni}
\end{array}
$$

$$
\dfrac{\dfrac{\vdots}{[x \mapsto^{\omega} \mathit{Nat}] \rhd^{\omega} [x \mapsto^{\omega} \mathit{Nat}]} \qquad \dfrac{\vdots}{[x \mapsto^{\omega} \mathit{Nat}, y \mapsto^{1} \mathit{Nat}] \vdash_{\mathsf{UA}} x + y :^{1} \mathit{Nat}}}{\dfrac{[x \mapsto^{\omega} \mathit{Nat}] \vdash_{\mathsf{UA}} \lambda y.\, x + y :^{\omega} \mathit{Nat}^{1} \to \mathit{Nat}^{1}}{\vdots}}
$$

Where are we?

- An annotated type system for usage analysis.
- Judgements of the form $\widehat{\Gamma} \vdash_{\mathsf{UA}} t :^{\varphi} \widehat{\sigma}$.
- Auxiliary judgement for context splitting: $\widehat{\Gamma} \sim_{\mathsf{UA}} \widehat{\Gamma}_1 \bowtie \widehat{\Gamma}_2$.
- Auxiliary judgement for containment: $\widehat{\Gamma} \triangleright^{\varphi} \widehat{\Gamma}_{11}$.

Applications

- Verification: type checking destructive updates (uniqueness typing).
- Optimization: avoiding unnecessary closure updates (sharing analysis).

5. Type checking destructive updates

Construct for destructive updates

- To demonstrate how the analysis can be used to perform uniqueness typing, we extend the language with a simple construct for destructive updates.

$$t \quad ::= \quad \cdots \mid x@t$$

- Meaning: update x with t.
- Can be formalized with a semantics that explicitly models memory usage.
- See Hage and Holdermans (PEPM 2008).

Typing rule for updates

- Require that updated object is unique.

$$\frac{\widehat{\Gamma}(x) =^{1} \widehat{\sigma}_0 \quad \widehat{\Gamma} \vdash_{\mathsf{UA}} t :^{\varphi} \widehat{\sigma}}{\widehat{\Gamma} \vdash_{\mathsf{UA}} x@t :^{\varphi} \widehat{\sigma}}$$

- Then: show that a program with updates has the same meaning as the same program with all updates removed.

6. Avoiding unnecessary closure updates

Generating use-once closures

- To avoid unnecessary closure updates, we compile to a target language that distinguishes between closures that can be used at most once and closures that can be used more than once.
- For each let-binding we indicate what kind of closure needs to be constructed.
- We make sure that closures are only created at let-bindings.

$$\widehat{t} \quad \in \quad \widehat{\mathbf{Tm}} \qquad \text{annotated terms}$$

$$\widehat{t} \quad ::= \quad \cdots \mid \widehat{t}_1\ x \mid \mathbf{let}\ x =^{\varphi} \widehat{t}_1\ \mathbf{in}\ \widehat{t}_2\ \mathbf{ni} \mid \cdots$$

- We equip the target language with a semantics that makes memory usage explicit and renders use-once closures inaccessible after their first use.

Target language: examples

$$\textbf{let } z =^{1} 2+3$$
$$\textbf{in } (\lambda x.\, x+1)\ z$$
$$\textbf{ni}$$

$$\textbf{let } z =^{\omega} 2+3$$
$$\textbf{in } (\lambda x.\, x+x)\ z$$
$$\textbf{ni}$$

Translation

- We write $\mathcal{T} :: \widehat{\Gamma} \vdash_{\mathsf{UA}} t :^{\varphi} \widehat{\sigma}$ to indicate that $\mathcal{T}$ is a proof tree for $\widehat{\Gamma} \vdash_{\mathsf{UA}} t :^{\varphi} \widehat{\sigma}$.
- Next, we define a translation $[\![-]\!]$ from proof trees to target terms.
- For example:

$$\left[\!\!\left[\frac{\begin{array}{c} \mathcal{T}_0 :: \widehat{\Gamma} \sim_{\mathsf{UA}} \widehat{\Gamma}_1 \bowtie \widehat{\Gamma}_2 \\ \mathcal{T}_1 :: \widehat{\Gamma}_1 \vdash_{\mathsf{UA}} t_1 :^{\varphi_1} \widehat{\sigma}_1 \\ \mathcal{T}_2 :: \widehat{\Gamma}_2[x \mapsto^{\varphi_1} \widehat{\sigma}_1] \vdash_{\mathsf{UA}} t_2 :^{\varphi} \widehat{\tau} \end{array}}{\widehat{\Gamma} \vdash_{\mathsf{UA}} \mathbf{let}\ x = t_1\ \mathbf{in}\ t_2\ \mathbf{ni} :^{\varphi} \widehat{\tau}} \right]\!\!\right] = \mathbf{let}\ x =^{\varphi_1} [\![\mathcal{T}_1]\!]\ \mathbf{in}\ [\![\mathcal{T}_2]\!]\ \mathbf{ni}$$

- Then, show that each translated program evaluates to the value of the original program.

7. Subeffecting

Lack of modularity

$$\textbf{let } x = 2 + 3$$
$$\textbf{in } (\lambda x.\, x + 1)\; x$$
$$\textbf{ni}$$

$$x :^{1} \mathit{Nat}$$

$$\textbf{let } x = 2 + 3$$
$$\textbf{in } (\lambda x.\, x + x)\; x$$
$$\textbf{ni}$$

$$x :^{\omega} \mathit{Nat}$$

☞ Use of x in body determines its usage annotation.

Poisoning

$$
\begin{array}{l}
\textbf{let } id = \lambda x.\, x \\
\textbf{in } \ \textbf{let } y = 2 + 3 \\
\quad\quad \textbf{in } \ \textbf{let } z = 5 \\
\quad\quad\quad\quad \textbf{in } \ id\ y + id\ z + z \\
\quad\quad\quad\quad \textbf{ni} \\
\quad\quad \textbf{ni} \\
\textbf{ni}
\end{array}
$$

- z is used more than once: hence, $z :^{\omega} Nat$.
- id is applied to z: hence, $id :^{\omega} Nat^{\omega} \to Nat^{\omega}$.
  (Or $id :^{\omega} \forall \alpha.\, \alpha^{\omega} \to \alpha^{\omega}$.)
- id is applied to y: hence, $y :^{\omega} Nat$.
- But y is used **only once!!**

Who's to blame?

- Recall the rule for function application:

$$\frac{\widehat{\Gamma} \sim_{\mathsf{UA}} \widehat{\Gamma}_1 \bowtie \widehat{\Gamma}_2 \quad \widehat{\Gamma}_1 \vdash_{\mathsf{UA}} t_1 :^{\varphi_1} \widehat{\tau}_2{}^{\varphi_2} \to \widehat{\tau}^{\varphi} \quad \widehat{\Gamma}_2 \vdash_{\mathsf{UA}} t_2 :^{\varphi_2} \widehat{\tau}_2}{\widehat{\Gamma} \vdash_{\mathsf{UA}} t_1\ t_2 :^{\varphi} \widehat{\tau}}$$

- Argument annotation φ_2 should match the annotation on the function domain.
- But in uniqueness typing, it's safe to bind a 1-annotated argument to an ω-annotated function parameter.
- Likewise, in sharing analysis, it's safe to bind an ω-annotated argument to a 1-annotated function parameter.

Ordering on annotations

Partial order on $\mathbf{Ann}$ with $1 \sqsubset \omega$:

$$\frac{}{1 \sqsubseteq \varphi}$$

$$\frac{}{\varphi \sqsubseteq \omega}$$

Subeffecting: uniqueness typing

- From our generic usage analysis we can derive a system that is specific for uniqueness typing.
- Judgements of the form $\widehat{\Gamma} \vdash_{\mathsf{UT}} t :^{\varphi} \widehat{\sigma}$.
- Same rules as before.
- New rule for subeffecting:

$$\frac{\widehat{\Gamma} \vdash_{\mathsf{UT}} t :^{\varphi_0} \widehat{\sigma} \qquad \varphi_0 \sqsubseteq \varphi}{\widehat{\Gamma} \vdash_{\mathsf{UT}} t :^{\varphi} \widehat{\sigma}}$$

Example

- Let $\widehat{\Gamma} = [f \mapsto^{1} (Nat^{\omega} \to Nat^{1}), x \mapsto^{1} Nat]$.
- For example: $f = \lambda x.\, x + x$ and $x = 2 + 3$.

$$\dfrac{\widehat{\Gamma} \sim_{\mathsf{UA}} \widehat{\Gamma}_1 \bowtie \widehat{\Gamma}_2 \qquad \dfrac{\widehat{\Gamma}_1(f) =^{1} Nat^{\omega} \to Nat^{1}}{\widehat{\Gamma}_1 \vdash_{\mathsf{UT}} f :^{1} Nat^{\omega} \to Nat^{1}} \qquad \dfrac{\dfrac{\widehat{\Gamma}_2(x) =^{1} Nat}{\widehat{\Gamma}_2 \vdash_{\mathsf{UT}} x :^{1} Nat} \quad 1 \sqsubseteq \omega}{\widehat{\Gamma}_2 \vdash_{\mathsf{UT}} x :^{\omega} Nat}}{\widehat{\Gamma} \vdash_{\mathsf{UT}} f\ x :^{1} Nat}$$

$\widehat{\Gamma}_1 = [f \mapsto^{1} (Nat^{\omega} \to Nat^{1})]$ and $\widehat{\Gamma}_2 = [x \mapsto^{1} Nat]$

Subeffecting: sharing analysis

- We can also derive a system that is specific for sharing analysis.
- Judgements of the form $\widehat{\Gamma} \vdash_{\mathsf{SA}} t :^{\varphi} \widehat{\sigma}$.
- Same rules as in the generic analysis.
- Again, a new rule for subeffecting:

$$\frac{\widehat{\Gamma} \vdash_{\mathsf{UT}} t :^{\varphi_0} \widehat{\sigma} \qquad \varphi \sqsubseteq \varphi_0}{\widehat{\Gamma} \vdash_{\mathsf{UT}} t :^{\varphi} \widehat{\sigma}}$$

Example

- Let $\widehat{\Gamma} = [f \mapsto^{1} (Nat^{1} \rightarrow Nat^{1}), x \mapsto^{\omega} Nat]$.
- For example: $f = \lambda x.\, x + 1$ and $x = 2 + 3$.

$$\dfrac{\widehat{\Gamma} \sim_{\mathsf{UA}} \widehat{\Gamma}_1 \bowtie \widehat{\Gamma}_2 \qquad \dfrac{\widehat{\Gamma}_1(f) =^{1} Nat^{1} \rightarrow Nat^{1}}{\widehat{\Gamma}_1 \vdash_{\mathsf{SA}} f :^{1} Nat^{1} \rightarrow Nat^{1}} \qquad \dfrac{\dfrac{\widehat{\Gamma}_2(x) =^{\omega} Nat}{\widehat{\Gamma}_2 \vdash_{\mathsf{SA}} x :^{\omega} Nat} \quad 1 \sqsubseteq \omega}{\widehat{\Gamma}_2 \vdash_{\mathsf{SA}} x :^{1} Nat}}{\widehat{\Gamma} \vdash_{\mathsf{SA}} f\ x :^{1} Nat}$$

$\widehat{\Gamma}_1 = [f \mapsto^{1} (Nat^{1} \rightarrow Nat^{1}), x \mapsto^{\omega} Nat]$ and $\widehat{\Gamma}_2 = [x \mapsto^{\omega} Nat]$

Keeping the analysis generic

- Define the inverse partial order $(\mathbf{Ann}, \sqsupseteq)$ with $\omega \sqsupset 1$.
- Let $\diamond$ range over the two partial orders:

$\diamond$	$\in$	$\mathbf{Ord} = \{\sqsubseteq, \sqsupseteq\}$	partial orders

- Parameterize the judgements of the generic analysis with a partial order $\diamond$:

$$\frac{\widehat{\Gamma} \vdash^{\diamond}_{\mathsf{UA}} t :^{\varphi_0} \widehat{\sigma} \qquad \varphi_0 \diamond \varphi}{\widehat{\Gamma} \vdash^{\diamond}_{\mathsf{UA}} t :^{\varphi} \widehat{\sigma}}$$

Instantiation

Uniqueness typing:

$$\frac{\widehat{\Gamma} \vdash^{\sqsubseteq}_{\mathsf{UA}} t :^{\varphi} \widehat{\sigma}}{\widehat{\Gamma} \vdash_{\mathsf{UT}} t :^{\varphi} \widehat{\sigma}}$$

Sharing analysis:

$$\frac{\widehat{\Gamma} \vdash^{\sqsupseteq}_{\mathsf{UA}} t :^{\varphi} \widehat{\sigma}}{\widehat{\Gamma} \vdash_{\mathsf{SA}} t :^{\varphi} \widehat{\sigma}}$$

8. Polyvariance

What about modularity?

- Idea: independent from its use sites, can we assign each function its "most flexible" type:
- For uniqueness analysis:

$$\begin{array}{ll} \lambda x.\, x + 1 & :^{\omega}\ Nat^{\omega} \rightarrow Nat^{1} \\ \lambda x.\, x & :^{\omega}\ Nat^{\omega} \rightarrow Nat^{\omega} \end{array}$$

- For sharing analysis:

$$\begin{array}{ll} \lambda x.\, x + 1 & :^{\omega}\ Nat^{1} \rightarrow Nat^{\omega} \\ \lambda x.\, x & :^{\omega}\ Nat^{\mathbf{??}} \rightarrow Nat^{\mathbf{??}} \end{array}$$

Polyvariance

- Allow types to be polymorphic in their annotations.
- For uniqueness analysis:

$$\begin{array}{lll} \lambda x.\, x+1 & :^{\omega} \forall\beta_1.\forall\beta_2. & Nat^{\beta_1} \to Nat^{\beta_2} \\ \lambda x.\, x & :^{\omega} \forall\beta. & Nat^{\beta} \to Nat^{\beta} \end{array}$$

- For sharing analysis:

$$\begin{array}{lll} \lambda x.\, x+1 & :^{\omega} \forall\beta_1.\forall\beta_2. & Nat^{\beta_1} \to Nat^{\beta_2} \\ \lambda x.\, x & :^{\omega} \forall\beta. & Nat^{\beta} \to Nat^{\beta} \end{array}$$

9. Subeffect qualifiers

How to capture all valid types?

In uniqueness typing (with subeffecting):

$$\lambda x.\, x :^{\omega} \forall \alpha.\, \alpha^{1} \to \alpha^{1}$$
$$\lambda x.\, x :^{\omega} \forall \alpha.\, \alpha^{1} \to \alpha^{\omega}$$
$$\lambda x.\, x :^{\omega} \forall \alpha.\, \alpha^{\omega} \to \alpha^{\omega}$$

In sharing analysis (with subeffecting):

$$\lambda x.\, x :^{\omega} \forall \alpha.\, \alpha^{1} \to \alpha^{1}$$
$$\lambda x.\, x :^{\omega} \forall \alpha.\, \alpha^{\omega} \to \alpha^{1}$$
$$\lambda x.\, x :^{\omega} \forall \alpha.\, \alpha^{\omega} \to \alpha^{\omega}$$

Which polyvariant type captures all valid types?

$\forall \alpha.\, \forall \beta.\quad \alpha^{\beta} \to \alpha^{\beta}$	(not general enough)
$\forall \alpha.\, \forall \beta_1.\, \forall \beta_2.\, \alpha^{\beta_1} \to \alpha^{\beta_2}$	(too general)

Poisoning (again)

$$
\begin{array}{l}
\textbf{let } h = \lambda f.\,\lambda x.\,\lambda y.\,f\ x + f\ y \\
\textbf{in } \ \textbf{let } g = \lambda z.\,z + 1 \\
\qquad \textbf{in } \ \textbf{let } u = 2 + 3 \\
\qquad\qquad \textbf{in } \ \textbf{let } v = 5 + 7 \\
\qquad\qquad\qquad \textbf{in } \ h\ g\ u\ v + v \\
\qquad\qquad\qquad \textbf{ni} \\
\qquad\qquad \textbf{ni} \\
\qquad \textbf{ni} \\
\textbf{ni}
\end{array}
$$

- Let $h :^{1} \forall\beta.\,(Nat^{\beta} \rightarrow Nat^{1})^{\omega} \rightarrow (Nat^{\beta} \rightarrow (Nat^{\beta} \rightarrow Nat^{1})^{1})^{1}$.
- v is used more than once, hence: $v :^{\omega} Nat$.
- But then, in the call to h, β is instantiated to ω.
- For sharing analysis, this means that $u :^{\omega} Nat$.
- But u is used **only once!!**

Qualified types

- To gain accuracy, we can store subeffecting conditions in type schemes.
- Qualified types are a generalization of Haskell's type classes that allow constraints to be incorporated in types.
- Elegant and well-established theory: see Jones (ESOP 1992).

$$\lambda x.\, x :^{\omega} \forall \alpha.\, \forall \beta_1.\, \forall \beta_2.\, \beta_1 \diamond \beta_2 \Rightarrow \alpha^{\beta_1} \to \alpha^{\beta_2}$$

Example revisited

$$
\begin{array}{l}
\textbf{let } h = \lambda f.\lambda x.\lambda y.\, f\ x + f\ y \\
\textbf{in}\ \ \textbf{let } g = \lambda z.\, z + 1 \\
\quad\ \ \textbf{in}\ \ \textbf{let } u = 2 + 3 \\
\qquad\quad \textbf{in}\ \ \textbf{let } v = 5 + 7 \\
\qquad\qquad\ \ \textbf{in}\ \ h\ g\ u\ v + v \\
\qquad\qquad\ \ \textbf{ni} \\
\qquad\quad \textbf{ni} \\
\quad\ \ \textbf{ni} \\
\textbf{ni}
\end{array}
$$

- Sharing analysis.
- Let $h :^1 \forall \beta_1\ \beta_2\ \beta_3.\ \beta_2 \sqsupseteq \beta_1 \Rightarrow \beta_3 \sqsupseteq \beta_1 \Rightarrow (\mathit{Nat}^{\beta_1} \to \mathit{Nat}^1)^\omega \to (\mathit{Nat}^{\beta_2} \to (\mathit{Nat}^{\beta_3} \to \mathit{Nat}^1)^1)^1$.
- v is used more than once, hence: $v :^\omega \mathit{Nat}$.
- So, in the call to h, β_3 is instantiated to ω.
- Still, the constraints are satisfied if $\beta_1 = \beta_2 = 1$.
- Hence, we can have $u :^1 \mathit{Nat}$.

Principal types

Most general types can sometimes be a bit intimidating.

$$\begin{array}{l}\lambda f.\lambda x.\lambda y.f\ x+f\ y: \\ \quad \forall\alpha.\forall\beta_1.\forall\beta_2.\forall\beta_3.\forall\beta_4.\forall\beta_5.\forall\beta_6.\forall\beta_7.\forall\beta_8. \\ \qquad \beta_3 \diamond \beta_1 \Rightarrow \beta_4 \diamond \beta_1 \Rightarrow \beta_7 \sqsubseteq \beta_3 \Rightarrow \\ \qquad (\alpha^{\beta_1} \to Nat^{\beta_2})^{\omega} \to (\alpha^{\beta_3} \to (\alpha^{\beta_4} \to Nat^{\beta_5})^{\beta_7})^{\beta_8}\end{array}$$

10. Properties of type systems (Metatheory)

Subject reduction

- If an expression has type τ, then the value it evaluates to also has type τ.
- Type preservation is a bit weaker: every evaluation step keeps the result well-typed.
  - But the types may change

Conservative extension

- If a program can be typed, then it can be analyzed.
- If a program can be analyzed, erasing the annotations from the proof tree gives the proof tree for the type system.

Safety/Soundness

- In the underlying type system: well-typed programs do not go wrong.
- In the annotated type system: acting on the optimisations implied by the annotations does not make evaluation go wrong.
- Usually, the semantics must be changed slightly to observe this.
- In the case of usage analysis:
  - Distinguish between 1-annotated and w-annotated thunks.
  - Remove the 1-annotated thunks from the heap when they have been used (once).
  - Show that you do never need to access something that was removed from the heap.

Progress

- Only with respect to small-step semantics.
- Evaluation of a well-typed term never gets stuck.

Completeness

- Usually the analysis is not complete
  - Some never-go-wrong expressions cannot be typed.
  - Static analysis is approximate.
- Still, we do sometimes establish completeness.
- Consider an analysis that generates constraints to capture the analysis.
- And build a solver to find a solution to the constraints.
- We want that solver to be
  - sound: the solution it computes is a solution
  - complete: if a set of constraints has a solution, the solver should find it (or a better solution).

Principality

- We prefer the analysis to provide a best solution,
- from which all other solutions can be derived.
- Depends very much on the expressivity of your types: $\lambda x.\, x$ may have type $\mathit{Nat} \to \mathit{Nat}$ or $\mathit{Bool} \to \mathit{Bool}$ if we do not allow type variables in types.
- Neither is better than the other.
- Principality allows to solve constraints, have the result be a principal type, and forget the constraints from then on.
- There is never a need to re-analyze: the principal type says all.
- Not to be confused with principal typings.