Data Mining 2013
Bayesian Networks (1)

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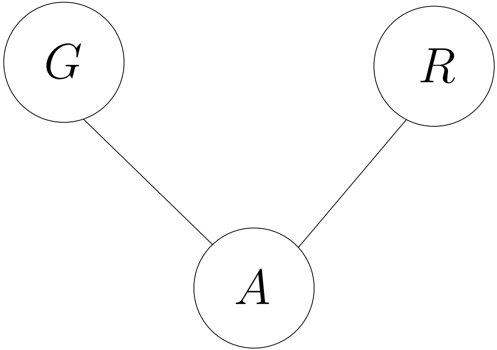
Do you like noodles?

Race	Gender	Do you like noodles? Yes	No
Black	Male	32	86
	Female	35	121
White	Male	61	73
	Female	42	70

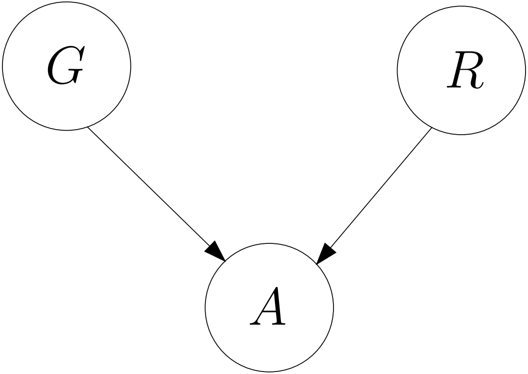
Do you like noodles? Undirected



$$G \perp\!\!\!\perp R \mid A$$

Strange: Gender and Race are prior to Answer, but this model says they are independent *given* Answer!

Do you like noodles? Directed



$$G \perp\!\!\!\perp R$$

Gender and Race are marginally independent (but *dependent* given Answer).

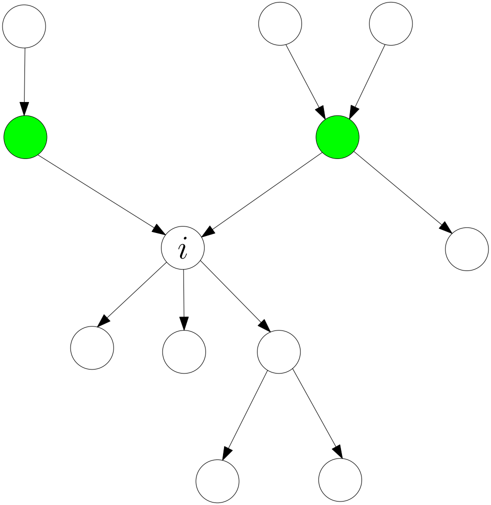
Directed Independence Graphs

$G = (K, E)$, K is a set of vertices and E is a set of edges with *ordered* pairs of vertices.

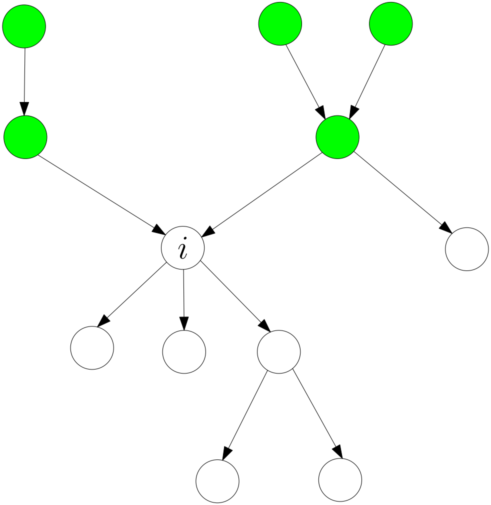
- No directed cycles (DAG)
- parent/child
- ancestor/descendant
- ancestral set

Because G is a DAG, there exists a *complete ordering* of the vertices that is respected in the graph (edges point from lower ordered to higher ordered nodes).

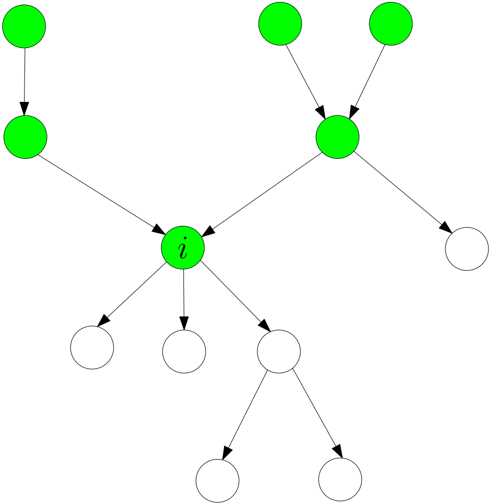
Parents Of Node i: pa(i)



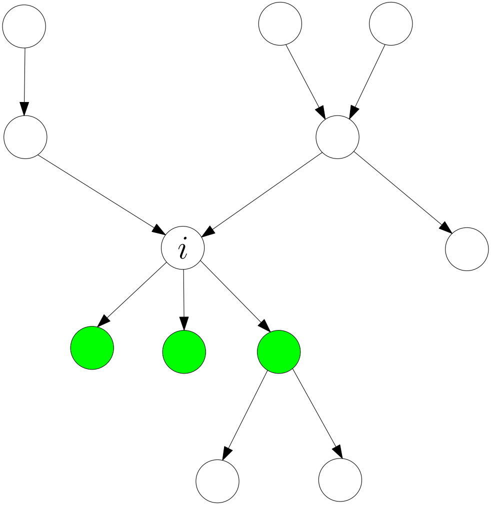
Ancestors Of Node i: an(i)



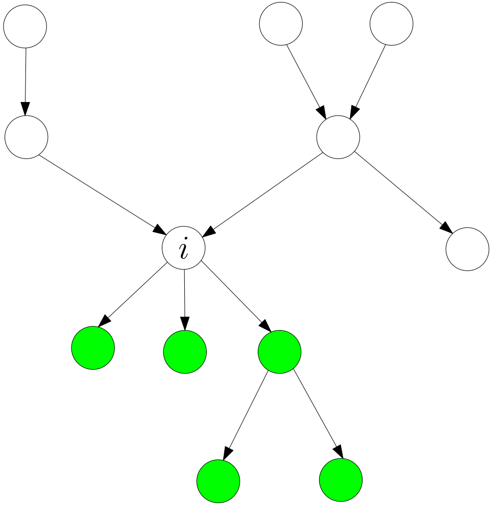
Ancestral Set Of Node i: $\text{an}^+(i)$



Children Of Node i: ch(i)



Descendants Of Node i: de(i)



Construction of DAG

Suppose that *prior* knowledge tells us the variables can be labeled $X_1, X_2, \ldots, X_k$ such that X_i is prior to X_{i+1}.
(for example: causal or temporal ordering)

Corresponding to this ordering we can use the product rule to factorize the joint distribution of $X_1, X_2, \ldots, X_k$ as

$$P(X) = P(X_1)P(X_2 \mid X_1) \cdots P(X_k \mid X_{k-1}, X_{k-2}, \ldots, X_1)$$

This is an identity of probability theory, no independence assumption have been made yet!

Constructing a DAG from pairwise independencies

In constructing a DAG, an arrow is drawn from i to j, where $i < j$, unless $P(X_j \mid X_{j-1}, \ldots, X_1)$ does not depend on X_i, in other words, unless

$$j \perp\!\!\!\perp i \mid \{1, \ldots, j\} \setminus \{i, j\}$$

More loosely

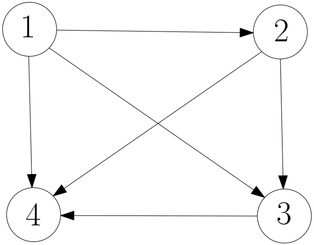
$$j \perp\!\!\!\perp i \mid \text{prior variables}$$

Compare this to pairwise independence

$$j \perp\!\!\!\perp i \mid \text{rest}$$

in undirected independence graphs.

Construction Of DAG

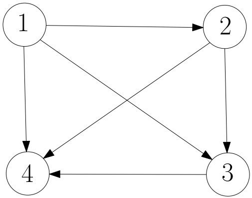


$$P(X) = P(X_4|X_1, X_2, X_3)P(X_3|X_1, X_2)P(X_2|X_1)P(X_1)$$

Suppose the following independencies are given:

1. $X_1 \perp\!\!\!\perp X_2$
2. $X_4 \perp\!\!\!\perp X_3|(X_1, X_2)$
3. $X_1 \perp\!\!\!\perp X_3|X_2$

Construction Of DAG

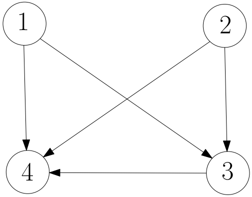


$$P(X) = P(X_4|X_1, X_2, X_3)P(X_3|X_1, X_2)\underbrace{P(X_2|X_1)}_{P(X_2)}P(X_1)$$

1. If $X_1 \perp\!\!\!\perp X_2$, then $P(X_2|X_1) = P(X_2)$.

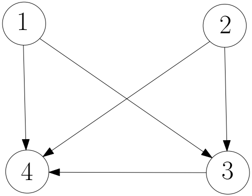
The edge $1 \rightarrow 2$ is removed.

Construction Of DAG



$$P(X) = P(X_4|X_1, X_2, X_3)P(X_3|X_1, X_2)P(X_2)P(X_1)$$

Construction Of DAG

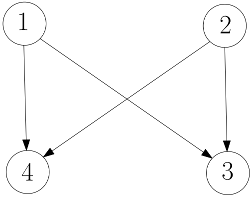


$$P(X) = \underbrace{P(X_4|X_1, X_2, X_3)}_{P(X_4|X_1,X_2)} P(X_3|X_1, X_2)P(X_2)P(X_1)$$

2. If $X_4 \perp\!\!\!\perp X_3|(X_1, X_2)$, then $P(X_4|X_1, X_2, X_3) = P(X_4|X_1, X_2)$.

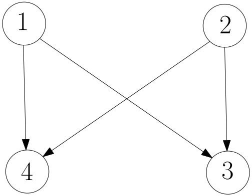
The edge $3 \rightarrow 4$ is removed.

Construction Of DAG



$$P(X) = P(X_4|X_1, X_2)P(X_3|X_1, X_2)P(X_2)P(X_1)$$

Construction Of DAG

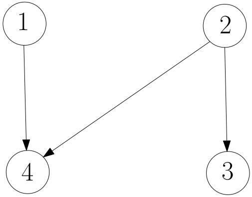


$$P(X) = P(X_4|X_1, X_2) \underbrace{P(X_3|X_1, X_2)}_{P(X_3|X_2)} P(X_2)P(X_1)$$

3. If $X_1 \perp\!\!\!\perp X_3|X_2$, then $P(X_3|X_1, X_2) = P(X_3|X_2)$

The edge $1 \rightarrow 3$ is removed.

Construction Of DAG



$$P(X) = P(X_4|X_1, X_2)P(X_3|X_2)P(X_2)P(X_1)$$

Joint density of Bayesian Network

We can write the joint density more elegantly as

$$P(X_1, \ldots, X_k) = \prod_{i=1}^{k} P(X_i \mid X_{pa(i)})$$

Independence Properties of DAGs: Moral Graph

Can we infer other/stronger independence statements from the directed graph like we did using separation in the undirected graphical models?

- d-separation (Pearl)
- make moral graph and use separation

Given a DAG $G = (K, E)$ we construct the moral graph G^m by marrying parents, and deleting directions, that is,

1. For each $i \in K$, we connect all vertices in $\text{pa}(i)$ with undirected edges.
2. We replace all directed edges in E with undirected ones.

Independence Properties of DAGs: Moral Graph

The directed independence graph G possesses the conditional independence properties of its associated moral graph G^m. Why? We have the factorisation:

$$P(X) = \prod_{i=1}^{k} P(X_i \mid X_{pa(i)})$$
$$= \prod_{i=1}^{k} g_i(X_i, X_{pa(i)})$$

by setting $g_i(X_i, X_{pa(i)}) = P(X_i \mid X_{pa(i)})$.

Independence Properties of DAGs: Moral Graph

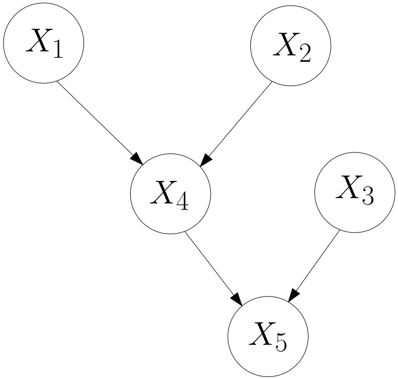
$$P(X) = \prod_{i=1}^{k} g_i(X_i, X_{pa(i)}) \qquad (1)$$

We thus have an expansion for the joint probability distribution in terms of functions $g(X_a)$ for $a = \{i\} \cup pa(i)$. Recall that $X \perp\!\!\!\perp Y \mid Z$ if and only if there exist functions g and h such that

$$P(x, y, z) = g(x, z)h(y, z)$$

By application of the factorisation criterion to the expansion (1), we can deduce all pairwise conditional independence statements of the form $i \perp\!\!\!\perp j \mid rest$.

Moralisation: Example



Moralisation: Example

This graph corresponds to the factorisation

$$\begin{aligned} P(X) &= P(X_1)P(X_2)P(X_3)P(X_4|X_1,X_2)P(X_5|X_3,X_4) \\ &= g_1(X_1)g_2(X_2)g_3(X_3)g_4(X_1,X_2,X_4)g_5(X_3,X_4,X_5) \end{aligned}$$

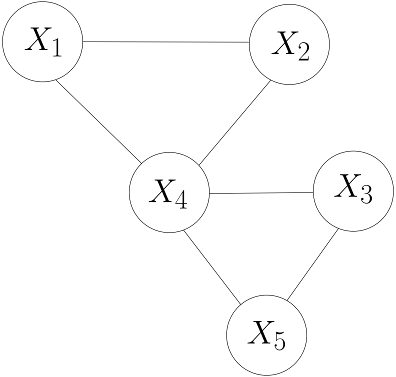
Log version:

$$\log P(X) = g_1^*(X_1) + g_2^*(X_2) + g_3^*(X_3) + g_4^*(X_1,X_2,X_4) + g_5^*(X_3,X_4,X_5),$$

where $g_i^*(X) = \log g_i(X)$.
We can read off the pairwise independencies:

$2 \perp\!\!\!\perp 3 \mid \textit{rest}$
$1 \perp\!\!\!\perp 3 \mid \textit{rest}$
$1 \perp\!\!\!\perp 5 \mid \textit{rest}$
$2 \perp\!\!\!\perp 5 \mid \textit{rest}$

Moralisation: Example



$\{i\} \cup pa(i)$ becomes a complete subgraph in the moral graph (by marrying all unmarried parents).

Moralisation Continued

Warning: the complete moral graph can obscure independencies!

To verify

$$i \perp\!\!\!\perp j \mid S$$

construct the moral graph on

$$A = \text{an}^+(\{i, j\} \cup S),$$

that is i, j, S and all their ancestors.

Moralisation Continued

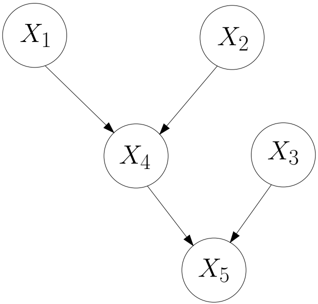
Since for $i \in A$, $pa(i) \in A$, we know that the joint distribution of X_A is given by

$$P(X_A) = \prod_{i \in A} P(X_i \mid X_{pa(i)})$$

which corresponds to the subgraph G_A of G.

1. This is a product of factors $P(X_i|X_{pa(i)})$, involving the variables $X_{\{i\} \cup pa(i)}$ only.
2. So it factorizes according to G_A^m, and thus the independence properties for undirected graphs apply.
3. So, if S separates i and j in G_A^m, then $i \perp\!\!\!\perp j \mid S$.

Moralisation Continued: example



Are X_3 and X_4 independent?

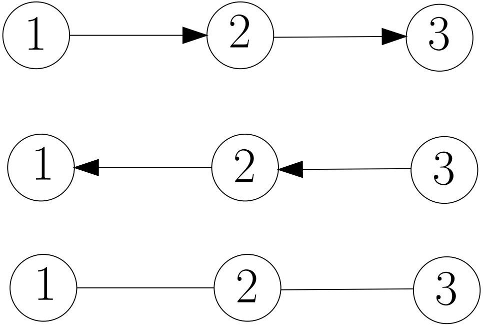
Are X_1 and X_3 independent?

Are X_3 and X_4 independent given X_5?

Equivalence

When no marrying of parents is required (there are no "v-structures"), then the independence properties of the directed graph are identical to those of its undirected version.

These three graphs express the same independence properties:



Learning Bayesian Networks: Overview

1. structure known, complete data
2. structure known, incomplete data
3. structure unknown, complete data
4. structure unknown, incomplete data: beyond the scope ...

Maximum Likelihood Estimation

Find value of unknown parameter(s) that maximize the probability of the observed data.

n independent observations on binary variable $X \in \{1, 2\}$. We observe $n(1)$ outcomes $X = 1$ and $n(2) = n - n(1)$ outcomes $X = 2$.
What is the maximum likelihood estimate of $p(1)$?
The likelihood function (probability of the data) is given by:

$$L = p(1)^{n(1)}(1 - p(1))^{n-n(1)}$$

Taking the log we get

$$\mathcal{L} = n(1) \log p(1) + (n - n(1)) \log(1 - p(1))$$

Maximum Likelihood Estimation

Take derivative with respect to $p(1)$, equate to zero, and solve for $p(1)$.

$$\frac{d\mathcal{L}}{dp(1)} = \frac{n(1)}{p(1)} - \frac{n - n(1)}{1 - p(1)} = 0,$$

since $\frac{d \log x}{dx} = \frac{1}{x}$ (where log is the natural logarithm).

Solving for $p(1)$, we get

$$p(1) = \frac{n(1)}{n},$$

i.e., the fraction of one's in the sample!

ML Estimation of Multinomial Distribution

Estimate the probabilities $p(1), p(2), \ldots, p(J)$ of getting outcomes $1, 2, \ldots, J$. If in n trials, we observe $n(1)$ outcomes of 1, $n(2)$ of 2, ..., $n(J)$ of J, then the obvious guess is to estimate

$$p(j) = \frac{n(j)}{n}, \qquad j = 1, \ldots, J$$

This is also the maximum likelihood estimate.

BN-Factorisation

For a given BN-DAG, the joint distribution factorises according to

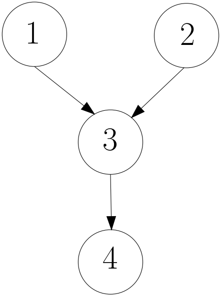
$$\prod_{i=1}^{k} p(X_i \mid X_{pa(i)})$$

So to specify the distribution we have to estimate the parameters

$$p(X_i \mid X_{pa(i)}) \qquad\qquad i = 1, 2, \ldots, k$$

The conditional distribution of each variable given its parents.

Example BN and Factorisation



$$P(X_1, X_2, X_3, X_4) = p_1(X_1)p_2(X_2)p_{3|12}(X_3|X_1, X_2)p_{4|3}(X_4|X_3)$$

Example BN: Parameters

$$P(X_1, X_2, X_3, X_4) = p_1(X_1)p_2(X_2)p_{3|12}(X_3|X_1, X_2)p_{4|3}(X_4|X_3)$$

Now we have to estimate the following parameters (X_4 ternary, rest binary):

$$\begin{array}{ll} p_1(1) & p_1(2) = 1 - p_1(1) \end{array}$$

$$\begin{array}{ll} p_2(1) & p_2(2) = 1 - p_2(1) \end{array}$$

$$\begin{array}{ll}
p_{3|1,2}(1|1,1) & p_{3|1,2}(2|1,1) = 1 - p_{3|1,2}(1|1,1) \\
p_{3|1,2}(1|1,2) & p_{3|1,2}(2|1,2) = 1 - p_{3|1,2}(1|1,2) \\
p_{3|1,2}(1|2,1) & p_{3|1,2}(2|2,1) = 1 - p_{3|1,2}(1|2,1) \\
p_{3|1,2}(1|2,2) & p_{3|1,2}(2|2,2) = 1 - p_{3|1,2}(1|2,2)
\end{array}$$

$$\begin{array}{lll}
p_{4|3}(1|1) & p_{4|3}(2|1) & p_{4|3}(3|1) = 1 - p_{4|3}(1|1) - p_{4|3}(2|1) \\
p_{4|3}(1|2) & p_{4|3}(2|2) & p_{4|3}(3|2) = 1 - p_{4|3}(1|2) - p_{4|3}(2|2)
\end{array}$$

Example Data Set

obs	X_1	X_2	X_3	X_4
1	1	1	1	1
2	1	1	1	1
3	1	1	2	1
4	1	2	2	1
5	1	2	2	2
6	2	1	1	2
7	2	1	2	3
8	2	1	2	3
9	2	2	2	3
10	2	2	1	3

Maximum Likelihood Estimation

obs	X_1	X_2	X_3	X_4
1	1	1	1	1
2	1	1	1	1
3	1	1	2	1
4	1	2	2	1
5	1	2	2	2
6	2	1	1	2
7	2	1	2	3
8	2	1	2	3
9	2	2	2	3
10	2	2	1	3

$$\hat{p}_1(1) = \frac{n(x_1 = 1)}{n} = \frac{5}{10} = \frac{1}{2}$$

Maximum Likelihood Estimation

obs	X_1	X_2	X_3	X_4
1	1	1	1	1
2	1	1	1	1
3	1	1	2	1
4	1	2	2	1
5	1	2	2	2
6	2	1	1	2
7	2	1	2	3
8	2	1	2	3
9	2	2	2	3
10	2	2	1	3

$$\hat{p}_2(1) = \frac{n(x_2 = 1)}{n} = \frac{6}{10}$$

Maximum Likelihood Estimation

obs	X_1	X_2	X_3	X_4
1	1	1	1	1
2	1	1	1	1
3	1	1	2	1
4	1	2	2	1
5	1	2	2	2
6	2	1	1	2
7	2	1	2	3
8	2	1	2	3
9	2	2	2	3
10	2	2	1	3

$$\hat{p}_{3|1,2}(1|1,1) = \frac{n(x_1 = 1, x_2 = 1, x_3 = 1)}{n(x_1 = 1, x_2 = 1)} = \frac{2}{3}$$

Maximum Likelihood Estimation

obs	X_1	X_2	X_3	X_4
1	1	1	1	1
2	1	1	1	1
3	1	1	2	1
4	1	2	2	1
5	1	2	2	2
6	2	1	1	2
7	2	1	2	3
8	2	1	2	3
9	2	2	2	3
10	2	2	1	3

$$\hat{p}_{3|1,2}(1|1,1) = \frac{n(x_1 = 1, x_2 = 1, x_3 = 1)}{n(x_1 = 1, x_2 = 1)} = \frac{2}{3}$$

ML Estimation of BN

The joint probability for n independent observations is

$$
\begin{aligned}
P(X^{(1)}, \ldots, X^{(n)}) &= \prod_{j=1}^{n} P(X^{(j)}) \\
&= \prod_{j=1}^{n} \prod_{i=1}^{k} p(X_i^{(j)} \mid X_{pa(i)}^{(j)})
\end{aligned}
$$

ML Estimation of BN

The likelihood function is thus given by

$$L = \prod_{i=1}^{k} \prod_{x_i, x_{pa(i)}} p(x_i \mid x_{pa(i)})^{n(x_i, x_{pa(i)})}$$

where $n(x_i, x_{pa(i)})$ is a count of the number of records with $X_i = x_i$, and $X_{pa(i)} = x_{pa(i)}$.

ML Estimation of BN

Taking the log of the likelihood, we get

$$\mathcal{L} = \sum_{i=1}^{k} \sum_{x_i, x_{pa(i)}} n(x_i, x_{pa(i)}) \log p(x_i \mid x_{pa(i)})$$

This is a collection of independent multinomial estimation problems.

The maximum likelihood estimate of $p(x_i \mid x_{pa(i)})$ is:

$$\hat{p}(x_i \mid x_{pa(i)}) = \frac{n(x_i, x_{pa(i)})}{n(x_{pa(i)})}$$

$n(x_i, x_{pa(i)})$: number of records in data with $X_i = x_i$ and $X_{pa(i)} = x_{pa(i)}$.

$n(x_{pa(i)})$: number of records in data with $X_{pa(i)} = x_{pa(i)}$.

Data Set and Likelihood

$$P(X_1, X_2, X_3, X_4) = p_1(X_1)p_2(X_2)p_{3|12}(X_3|X_1, X_2)p_{4|3}(X_4|X_3)$$

obs	X_1	X_2	X_3	X_4
1	1	1	1	1
2	1	1	1	1
3	1	1	2	1
4	1	2	2	1
5	1	2	2	2
6	2	1	1	2
7	2	1	2	3
8	2	1	2	3
9	2	2	2	3
10	2	2	1	3

$p_1(1)p_2(1)p_{3|12}(1|1,1)p_{4|3}(1|1)$

Data Set and Likelihood

$$P(X_1, X_2, X_3, X_4) = p_1(X_1)p_2(X_2)p_{3|12}(X_3|X_1, X_2)p_{4|3}(X_4|X_3)$$

obs	X_1	X_2	X_3	X_4
1	1	1	1	1
2	1	1	1	1
3	1	1	2	1
4	1	2	2	1
5	1	2	2	2
6	2	1	1	2
7	2	1	2	3
8	2	1	2	3
9	2	2	2	3
10	2	2	1	3

$p_1(1)p_2(1)p_{3|12}(1|1,1)p_{4|3}(1|1)$

$p_1(1)p_2(1)p_{3|12}(1|1,1)p_{4|3}(1|1)$

Data Set and Likelihood

$$P(X_1, X_2, X_3, X_4) = p_1(X_1)p_2(X_2)p_{3|12}(X_3|X_1, X_2)p_{4|3}(X_4|X_3)$$

obs	X_1	X_2	X_3	X_4	
1	1	1	1	1	$p_1(1)p_2(1)p_{3\|12}(1\|1, 1)p_{4\|3}(1\|1)$
2	1	1	1	1	$p_1(1)p_2(1)p_{3\|12}(1\|1, 1)p_{4\|3}(1\|1)$
3	1	1	2	1	$p_1(1)p_2(1)p_{3\|12}(2\|1, 1)p_{4\|3}(1\|2)$
4	1	2	2	1	
5	1	2	2	2	
6	2	1	1	2	
7	2	1	2	3	
8	2	1	2	3	
9	2	2	2	3	
10	2	2	1	3	

Data Set and Likelihood

$$P(X_1, X_2, X_3, X_4) = p_1(X_1)p_2(X_2)p_{3|12}(X_3|X_1, X_2)p_{4|3}(X_4|X_3)$$

obs	X_1	X_2	X_3	X_4	
1	1	1	1	1	$p_1(1)p_2(1)p_{3\|12}(1\|1,1)p_{4\|3}(1\|1)$
2	1	1	1	1	$p_1(1)p_2(1)p_{3\|12}(1\|1,1)p_{4\|3}(1\|1)$
3	1	1	2	1	$p_1(1)p_2(1)p_{3\|12}(2\|1,1)p_{4\|3}(1\|2)$
4	1	2	2	1	$p_1(1)p_2(2)p_{3\|12}(2\|1,2)p_{4\|3}(1\|2)$
5	1	2	2	2	
6	2	1	1	2	
7	2	1	2	3	
8	2	1	2	3	
9	2	2	2	3	
10	2	2	1	3	

Data Set and Likelihood

$$P(X_1, X_2, X_3, X_4) = p_1(X_1)p_2(X_2)p_{3|12}(X_3|X_1, X_2)p_{4|3}(X_4|X_3)$$

obs	X_1	X_2	X_3	X_4	
1	1	1	1	1	$p_1(1)p_2(1)p_{3|12}(1|1,1)p_{4|3}(1|1)$
2	1	1	1	1	$p_1(1)p_2(1)p_{3|12}(1|1,1)p_{4|3}(1|1)$
3	1	1	2	1	$p_1(1)p_2(1)p_{3|12}(2|1,1)p_{4|3}(1|2)$
4	1	2	2	1	$p_1(1)p_2(2)p_{3|12}(2|1,2)p_{4|3}(1|2)$
5	1	2	2	2	$p_1(1)p_2(2)p_{3|12}(2|1,2)p_{4|3}(2|2)$
6	2	1	1	2	
7	2	1	2	3	
8	2	1	2	3	
9	2	2	2	3	
10	2	2	1	3	

Data Set and Likelihood

$$P(X_1, X_2, X_3, X_4) = p_1(X_1)p_2(X_2)p_{3|12}(X_3|X_1, X_2)p_{4|3}(X_4|X_3)$$

obs	X_1	X_2	X_3	X_4	
1	1	1	1	1	$p_1(1)p_2(1)p_{3\|12}(1\|1,1)p_{4\|3}(1\|1)$
2	1	1	1	1	$p_1(1)p_2(1)p_{3\|12}(1\|1,1)p_{4\|3}(1\|1)$
3	1	1	2	1	$p_1(1)p_2(1)p_{3\|12}(2\|1,1)p_{4\|3}(1\|2)$
4	1	2	2	1	$p_1(1)p_2(2)p_{3\|12}(2\|1,2)p_{4\|3}(1\|2)$
5	1	2	2	2	$p_1(1)p_2(2)p_{3\|12}(2\|1,2)p_{4\|3}(2\|2)$
6	2	1	1	2	$p_1(2)p_2(1)p_{3\|12}(1\|2,1)p_{4\|3}(2\|1)$
7	2	1	2	3	$p_1(2)p_2(1)p_{3\|12}(2\|2,1)p_{4\|3}(3\|2)$
8	2	1	2	3	$p_1(2)p_2(1)p_{3\|12}(2\|2,1)p_{4\|3}(3\|2)$
9	2	2	2	3	$p_1(2)p_2(2)p_{3\|12}(2\|2,2)p_{4\|3}(3\|2)$
10	2	2	1	3	$p_1(2)p_2(2)p_{3\|12}(1\|2,2)p_{4\|3}(3\|1)$

The Likelihood Function

Estimate 10 probabilities in total.

Contribution of observation 1 to the likelihood:

$$L(1,1,1,1) = p_1(1)p_2(1)p_{3|1,2}(1|1,1)p_{4|3}(1|1)$$

Contribution of observation 3:

$$\begin{aligned} L(1,1,2,1) &= p_1(1)p_2(1)p_{3|1,2}(2|1,1)p_{4|3}(1|2) \\ &= p_1(1)p_2(1)(1 - p_{3|1,2}(1|1,1))p_{4|3}(1|2) \end{aligned}$$

Joint contribution of observation 1 and 3 is:

$$p_1(1)^2 p_2(1)^2 p_{3|1,2}(1|1,1)(1 - p_{3|1,2}(1|1,1))p_{4|3}(1|1)p_{4|3}(1|2)$$

For all observations

Likelihood function for all observations together:

$$
\begin{aligned}
L(\mathcal{D}) \quad = \quad & p_1(1)^5(1-p_1(1))^5 p_2(1)^6(1-p_2(1))^4 p_{3|1,2}(1|1,1)^2(1-p_{3|1,2}(1|1,1)) \\
& (1-p_{3|1,2}(1|1,2))^2 p_{3|1,2}(1|2,1)(1-p_{3|1,2}(1|2,1))^2 p_{3|1,2}(1|2,2) \\
& (1-p_{3|1,2}(1|2,2)) p_{4|3}(1|1)^2 p_{4|3}(2|1)(1-p_{4|3}(1|1)-p_{4|3}(2|1)) \\
& p_{4|3}(1|2)^2 p_{4|3}(2|2)(1-p_{4|3}(1|2)-p_{4|3}(2|2))^3
\end{aligned}
$$

Or in log form

$$
\begin{aligned}
\mathcal{L}(\mathcal{D}) \quad = \quad & 5\log p_1(1) + 5\log(1-p_1(1)) + 6\log p_2(1) + 4\log(1-p_2(1)) \\
& +2\log p_{3|1,2}(1|1,1) + \log(1-p_{3|1,2}(1|1,1)) \\
& +2\log(1-p_{3|1,2}(1|1,2)) + \log p_{3|1,2}(1|2,1) + 2\log(1-p_{3|1,2}(1|2,1)) \\
& +\log p_{3|1,2}(1|2,2) + \log(1-p_{3|1,2}(1|2,2)) \\
& +2\log p_{4|3}(1|1) + \log p_{4|3}(2|1) + \log(1-p_{4|3}(1|1)-p_{4|3}(2|1)) \\
& +2\log p_{4|3}(1|2) + \log p_{4|3}(2|2) + 3\log(1-p_{4|3}(1|2)-p_{4|3}(2|2))
\end{aligned}
$$

ML estimate of $p_{3|1,2}(1|1,1)$

Take partial derivative of $\mathcal{L}$ wrt $p = p_{3|1,2}(1|1,1)$:

$$\mathcal{L} = \ldots + 2\log p + \log(1-p) + \ldots$$
$$\frac{\partial \mathcal{L}}{\partial p} = \frac{2}{p} - \frac{1}{1-p}$$

Equate to zero and solve for p: $p = \frac{2}{3}$

For all observations

Likelihood function for all observations together:

$$
\begin{aligned}
L(\mathcal{D}) \quad = \quad & p_1(1)^5(1-p_1(1))^5 p_2(1)^6(1-p_2(1))^4 p_{3|1,2}(1|1,1)^2(1-p_{3|1,2}(1|1,1)) \\
& (1-p_{3|1,2}(1|1,2))^2 p_{3|1,2}(1|2,1)(1-p_{3|1,2}(1|2,1))^2 p_{3|1,2}(1|2,2) \\
& (1-p_{3|1,2}(1|2,2)) p_{4|3}(1|1)^2 p_{4|3}(2|1)(1-p_{4|3}(1|1)-p_{4|3}(2|1)) \\
& p_{4|3}(1|2)^2 p_{4|3}(2|2)(1-p_{4|3}(1|2)-p_{4|3}(2|2))^3
\end{aligned}
$$

Or in log form

$$
\begin{aligned}
\mathcal{L}(\mathcal{D}) \quad = \quad & 5\log p_1(1) + 5\log(1-p_1(1)) + 6\log p_2(1) + 4\log(1-p_2(1)) \\
& +2\log p_{3|1,2}(1|1,1) + \log(1-p_{3|1,2}(1|1,1)) \\
& +2\log(1-p_{3|1,2}(1|1,2)) + \log p_{3|1,2}(1|2,1) + 2\log(1-p_{3|1,2}(1|2,1)) \\
& +\log p_{3|1,2}(1|2,2) + \log(1-p_{3|1,2}(1|2,2)) \\
& +2\log p_{4|3}(1|1) + \log p_{4|3}(2|1) + \log(1-p_{4|3}(1|1)-p_{4|3}(2|1)) \\
& +2\log p_{4|3}(1|2) + \log p_{4|3}(2|2) + 3\log(1-p_{4|3}(1|2)-p_{4|3}(2|2))
\end{aligned}
$$