Coquet: A Coq library for verifying hardware

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SYNCHRON 2011

Representing circuits with predicates (or functions).

- Some definitions:

$$Xor(i_1, i_2, o) \triangleq (o = \neg(i_1 = i_2)) \qquad Not(i, o) \triangleq (o = \neg i)$$

- Adding structure:
- Correctness proof: entailment of a specification.

$$(\exists x, Xor(i_1, i_2, x) \wedge Not(x, o)) \implies (o = (i_1 = i_2))$$

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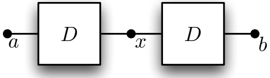

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Formalising circuits in proof assistants

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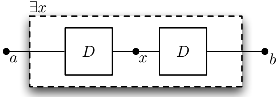

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Composition

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The good points of a shallow embedding

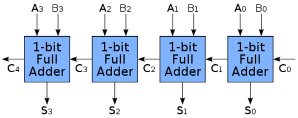
Representing circuits with predicates of the host language makes modelling of circuits easy.

- Use the binders of the theorem prover: $\forall$, $\exists$.
- Use recursion to define recursive structure:

```
let rec mux n (sel,a,b,out) = match n with
| 0   → ⊤
| S n → hd out = (if sel then hd a else hd b)
     ∧ mux n (sel,tl a, tl b, tl out)
```

Use lists to model bit-vectors. We have `a`, `b`, `out`: `bool list`.

The bad points of a shallow embedding



- Let's define a recursive adder.

  let rec adder n (a,b,cin,sum,cout) = match n with
  | 0 $\rightarrow \top$
  | S n $\rightarrow$ $\exists$ c. adder n (tl a, tl b, c, tl sum, cout)
  $\wedge$ add1 (hd a, hd b, cin, hd sum, c)

- Alternatively,

  let adder (a,b,cin,sum,cout) =
  let cout',sum' = List.fold_right2 (λ a b (c,res) $\rightarrow$ ...) a b (cin,[]) in
  sum = sum' $\wedge$ cout = cout'

Question

What is a circuit ?

Using a shallow-embedding, there is no way to:

- restrict the quantification on circuits;
- reason on the structure of the circuit in the proof assistant.

Move to a deep-embedding:

- define a data structure for circuits;
- define what's a circuit semantics (via an interpretation function);
- prove that a device implements a given specification.

Shallow-Embeddings vs Deep-Embedding

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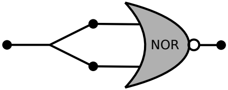
This talk

- Use Coq to embed a language for (synchronous) circuits
- Prove the functionnal correction of circuits

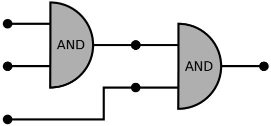
No currents, no delays



`Fork 2 ▷ Atom NOR`

`Ser 1 2 1 (Fork 2) (Atom NOR)`

Gate `Not` : `circuit 1 1`



`(And & One 1) ▷ And`

`Ser 3 2 1 (Par 2 1 1 1 AND (One 1)) AND`

Gate `And3` : `circuit 3 1`

Outline

1. Defining a deep-embedding of circuits
2. Recursive circuits
3. Sequential circuits: time and loops
4. Conclusion, perspectives and related works

A dependent type for circuits in Coq

- First version:

```
Inductive ℂ : nat → nat → Type := ...
```

- Examples:

$$\begin{array}{lcl} \texttt{Not} & : & \mathbb{C}\ 1\ 1 \\ \texttt{And3} & : & \mathbb{C}\ 3\ 1 \\ \texttt{Adder}\ n & : & \mathbb{C}\ (2n+1)\ (n+1) \end{array}$$

- Does not give much structure!

$$\begin{aligned} 2n+1 &= n+n+1 \\ &= n+1+n \\ &= 1+n+n \end{aligned}$$

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A better dependent type for circuits in Coq

- We use arbitrary types as indexes for the ports:

  $\texttt{Inductive}\ \mathbb{C}\ : \texttt{Type} \to \texttt{Type} \to \texttt{Type} := \ldots$

- For instance ($\mathbf{1}$ is the unit type, and $\oplus$ is disjoint-sum):

`Not`	:	$\mathbb{C}\ \mathbf{1}\ \mathbf{1}$	semantics$(\mathbf{1} \to \mathbb{B}) \to (\mathbf{1} \to \mathbb{B})$
`And3`	:	$\mathbb{C}\ (\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1})\ \mathbf{1}$	semantics $(\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1} \to \mathbb{B}) \to (\mathbf{1} \to \mathbb{B})$
`Adder` n	:	$\mathbb{C}\ (n \cdot \mathbf{1} \oplus n \cdot \mathbf{1} \oplus \mathbf{1})\ (n \cdot \mathbf{1} \oplus \mathbf{1})$	...

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- Note. The indices are tags, used to identify $\mathbf{1}$. (Can use any infinite type.)

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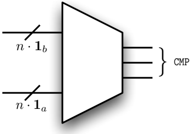
- Note. The indices are tags, used to identify $\mathbf{1}$. (Can use any infinite type.)
- Can use other types. `Compare` $n : \mathbb{C}\ (n \cdot \mathbf{1}_{\mathtt{a}} \oplus n \cdot \mathbf{1}_{\mathtt{b}})\ (\mathtt{CMP})$ where

  `Inductive CMP : Type := | Eq | Lt | Gt.`



Plugs

We use circuit combinators ($\&, \triangleright$).

- The information flow is implicit.
- Nameless setting: ports have to be duplicated and reordered using plugs.
- A plug is a circuit of type $\mathbb{C}\ n\ m$ ... defined as a map from m to n.
- Forbids short-circuits.

Example

$\mathbb{C}\ (n \oplus m)\ m$

$$\begin{array}{ccc} m & \to & n \oplus m \\ x & \mapsto & \texttt{inr}\ x \end{array}$$

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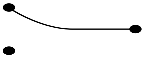
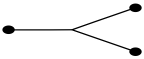


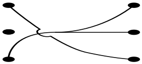
$\mathbb{C}\ (n \oplus m)\ m$

(b)



$\mathbb{C}\ n\ (n \oplus n)$

(c)



$\mathbb{C}\ (n \oplus m \oplus p)\ (p \oplus (n \oplus n))$

types must be read bottom-up

a) fun (x : m) ⇒ inr n x

b) fun (x : n ⊕ n) ⇒ match x with inl e ⇒ e | inr e ⇒ e end.

c) fun (x : p ⊕ (n ⊕ n)) ⇒ match x with
| inl ep ⇒ inr (n ⊕ m) ep
| inr (inl en) ⇒ inl p (inl m en)
| inr (inr en) ⇒ inl p (inl m en)

by proof-search

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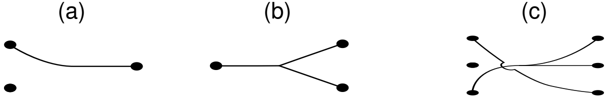

$\mathbb{C}\ (n \oplus m)\ m$ — $\mathbb{C}\ n\ (n \oplus n)$ — $\mathbb{C}\ (n \oplus m \oplus p)\ (p \oplus (n \oplus n))$

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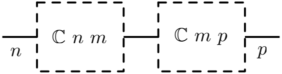
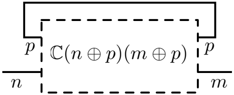
by proof-search

Abstract syntax

- Strongly typed syntax

```
Inductive ℂ : Type → Type → Type :=
| Atom : ∀ (n m : Type), atom n m → ℂ n m
| Plug : ∀ (n m : Type) (f : m → n), ℂ n m
| Ser  : ∀ (n m p : Type), ℂ n m → ℂ m p → ℂ n p
| Par  : ∀ (n m p q : Type), ℂ n p → ℂ m q →  ℂ (n ⊕ m) (p ⊕ q)
| Loop : ∀ (n m p : Type), ℂ (n ⊕ p) (m ⊕ p) → ℂ n m.
```





- Intrinsic approach: an alternative to syntax + typing judgement.

The semantics of a circuit

For a circuit of type $\mathbb{C}\ n\ m$, a relation between $n \to \mathbb{T}$ and $m \to \mathbb{T}$.

Rules

$$\text{KSer}\ \frac{x \vdash^n_m ins \bowtie middle \qquad y \vdash^m_p middle \bowtie outs}{x \rhd y \vdash^n_p ins \bowtie outs}$$

$$\text{KPar}\ \frac{x \vdash^n_p \texttt{left}\ ins \bowtie \texttt{left}\ outs \qquad y \vdash^m_q \texttt{right}\ ins \bowtie \texttt{right}\ outs}{x \& y \vdash^{n \oplus m}_{p \oplus q} ins \bowtie outs}$$

$$\text{KPlug}\ \frac{}{\texttt{Plug}\ f \vdash^n_m ins \bowtie \texttt{lift}\ f\ ins} \qquad \text{KLoop}\ \frac{x \vdash^{n \oplus p}_{m \oplus p} \text{app}\ ins\ r \bowtie \text{app}\ outs\ r}{\texttt{Loop}\ x \vdash^n_m ins \bowtie outs}$$

Parametric in the base doors, the type $\mathbb{T}$ and the semantics of the base doors.

Operations

```
Definition left n m : ((n ⊕ m) → 𝕋) → (n → 𝕋) := ...
Definition app n m : (n → 𝕋) → (m → 𝕋) → (n ⊕ m → 𝕋) := ...
Definition lift n m m (f : m → n) (ins : n → 𝕋) : (m → 𝕋) := ins ∘ f.
```

The need for abstraction

The semantics of a circuit defines precisely its behavior, but:

- is too precise (may leak some internal details);
- is a relation between two functions $n \to \mathbb{T}$ and $m \to \mathbb{T}$. (Example: $\mathbf{1} \oplus \mathbf{1} \to \mathbb{B}$...).

Use type isomorphisms as "lenses":

```
Class Iso (A B : Type) :={
 iso : A → B;
 uniso : B → A}.
```

```
Class Iso_Props {A B: Type} (I : Iso A B):= {
 iso_uniso : ∀ (x : B), iso (uniso x) = x;
 uniso_iso : ∀ (x : A), uniso (iso x) = x}.
```

Examples:

$$\iota_x \; \frac{}{\mathbf{1}_x \to \mathbb{T} \cong \mathbb{T}} \qquad \cdot \bullet \cdot \; \frac{A \to \mathbb{T} \cong \sigma \qquad B \to \mathbb{T} \cong \tau}{A \oplus B \to \mathbb{T} \cong (\sigma \times \tau)} \qquad \frac{A \to \mathbb{T} \cong \sigma}{n \cdot A \to \mathbb{T} \cong \mathtt{vector}\ \sigma\ \mathtt{n}}$$

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Putting it all together

We define type classes for abstractions (and modular proofs).

Useful for proof automation

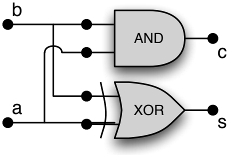
```
Context (n m N M : Type) (Rn : (n→ 𝕋) ≅ N) (Rm : (m→ 𝕋) ≅ M).

Class Realise (c : ℂ n m) (R : N → M → Prop) :=
realise: ∀ ins outs, c ⊢ⁿₘ ins ⋈ outs → R (Rn.iso ins) (Rm.iso outs).

Class Implement (c : ℂ n m) (f : N → M) :=
implement: ∀ ins outs, c ⊢ⁿₘ ins ⋈ outs → Rm.iso outs = f (Rn.iso ins).
```

"Up-to isomorphisms, a given circuit implements a given function."

A complete example

Definition HADD : $\mathbb{C}$ $(\mathbf{1}_a \oplus \mathbf{1}_b)$ $(\mathbf{1}_s \oplus \mathbf{1}_c)$:=
Fork 2 $(\mathbf{1}_a \oplus \mathbf{1}_b)$ ▷ (XOR a b s & AND a b c).

Definition hadd := λ (a,b).(a ⊗ b, a ∧ b)



Lemma HADD_Spec : Implement
$(\iota_a \bullet \iota_b)$
$(\iota_s \bullet \iota_c)$
HADD hadd.

I : $\mathbf{1}_a \oplus \mathbf{1}_b \rightarrow \mathbb{B}$, O : $\mathbf{1}_s \oplus \mathbf{1}_c \rightarrow \mathbb{B}$
H : HADD $\vdash^{\mathbf{1}_a \oplus \mathbf{1}_b}_{\mathbf{1}_s \oplus \mathbf{1}_c}$ I ⋈ O
=====================
@iso $(\iota_s \bullet \iota_c)$ O = hadd (@iso $(\iota_a \bullet \iota_b)$ I)

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M : $(\mathbf{1}_a \oplus \mathbf{1}_b) \oplus (\mathbf{1}_a \oplus \mathbf{1}_b) \rightarrow \mathbb{B}$
H0: iso M = (fun x ⇒ (x,x)) (iso I)
H1: iso (left O) = uncurry ⊗ (iso (left M))
H2: iso (right O)= uncurry ∧ (iso (right M))
============================
iso O = hadd (iso I)

I: $\mathbb{B} * \mathbb{B}$, O: $\mathbb{B} * \mathbb{B}$,
M : $(\mathbb{B} * \mathbb{B}) * (\mathbb{B} * \mathbb{B})$,
H0: M = (fun x ⇒ (x,x)) I
H1: fst O = uncurry ⊗ (fst M)
H2: snd O = uncurry ∧ (snd M)
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O = hadd I

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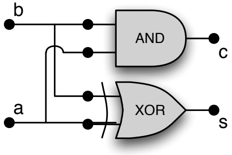

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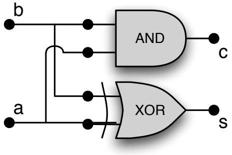

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H1: iso (left O) = uncurry ⊗ (iso (left M))
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iso O = hadd (iso I)

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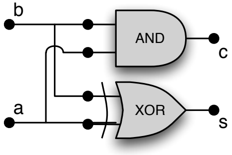
A complete example

Definition HADD : $\mathbb{C}$ $(\mathbf{1}_a \oplus \mathbf{1}_b)$ $(\mathbf{1}_s \oplus \mathbf{1}_c)$:=
Fork 2 $(\mathbf{1}_a \oplus \mathbf{1}_b)$ ▷ (XOR a b s & AND a b c).

Definition hadd := λ (a,b).(a ⊗ b, a ∧ b)



Lemma HADD_Spec : Implement
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I : $\mathbf{1}_a \oplus \mathbf{1}_b \to \mathbb{B}$, O : $\mathbf{1}_s \oplus \mathbf{1}_c \to \mathbb{B}$
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=====================
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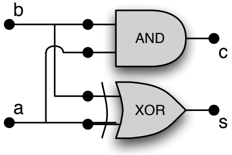

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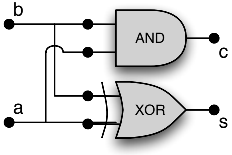

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H1: fst O = uncurry $\otimes$ (fst M)
H2: snd O = uncurry $\wedge$ (snd M)
==================
O = hadd I

Outline

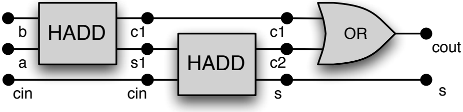
n-bit integers

Record $\mathbb{W}_n$:= mk_word {val : $\mathbb{Z}$; range: $0 \leq val < 2^n$}.

Definition repr n : Z → $\mathbb{W}_n$:= ...
Definition high n m : $\mathbb{W}_{(n+m)} \rightarrow \mathbb{W}_m$:= ...
Definition low n m : $\mathbb{W}_{(n+m)} \rightarrow \mathbb{W}_n$:= ...
Definition combine n m : $\mathbb{W}_n \rightarrow \mathbb{W}_m \rightarrow \mathbb{W}_{(n+m)}$:= ...
Definition carry_add n (x y : $\mathbb{W}_n$) (b: $\mathbb{B}$) : $\mathbb{W}_n * \mathbb{B}$:=
    let e := val x + val y + (if b then 1 else 0) in (e mod 2^n,$2^n \leq$ e)

Definition $\Phi^n_x : (n \cdot \mathbf{1}_x \rightarrow \mathbb{B}) \cong (\mathbb{W}_n)$:= ...

A 1-bit adder

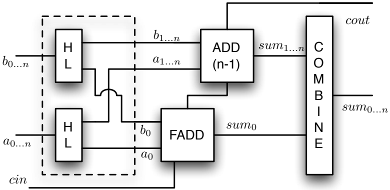



```
Context a b cin sum cout : string.
Program Definition FADD :
```
$\mathbb{C}$ ($\mathbf{1}_{cin} \oplus (\mathbf{1}_a \oplus \mathbf{1}_b)$) ($\mathbf{1}_{sum} \oplus \mathbf{1}_{cout}$):=
 (ONE $\mathbf{1}_{cin}$ & HADD a b "s1" "c1")
 ▷ ...
 ▷ (HADD cin "s1" sum "c2" & ONE $\mathbf{1}_{\text{"}c1\text{"}}$)
 ▷ ...
 ▷ (ONE $\mathbf{1}_{sum}$ & OR "c2" "c1" cout).

Instance FADD_1 : Implement
$(\iota_{\mathrm{cin}} \bullet (\iota_{\mathrm{a}} \bullet \iota_{\mathrm{b}}))$
$(\iota_{\mathrm{sum}} \bullet \iota_{\mathrm{cout}})$
FADD
(fun (c,(x,y)) $\Rightarrow$ $(x \oplus (y \oplus c), (x \wedge y) \vee c \wedge (x \oplus y))$).

Instance FADD_2 : Implement
$(\iota_{\mathrm{cin}} \bullet (\Phi^1_{\mathrm{a}} \bullet \Phi^1_{\mathrm{b}}))$
$(\Phi^1_{\mathrm{sum}} \bullet \iota_{\mathrm{cout}})$
FADD
(fun (c,(x,y)) $\Rightarrow$ carry_add 1 x y c).

A n-bit adder

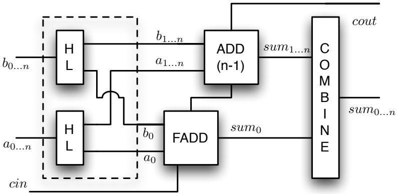



```
Program Fixpoint ADD cin a b cout sum n :
 ℂ (1_cin ⊕ n · 1_a ⊕ n · 1_b) (n · 1_sum ⊕ 1_cout) :=
match n with
   | 0 ⇒ ...
   | S p ⇒ ... ▷ (ONE (1_cin) & HIGHLOWS a b 1 p)
   ▷ ... ▷ (FADD a b cin sum "c"  & ONE (p · 1_a ⊕ p · 1_b))
   ▷ ... ▷ (ONE (1_sum) & ADD "c" a b cout s p)
   ▷ ... ▷ COMBINE sum 1 p & ONE (1_cout)
end.
```

```
Lemma add_parts n m (xH yH: 𝕎_m) (xL yL : 𝕎_n) cin:
let (sumL,middle) := carry_add n xL yL cin in
let (sumH,cout) := carry_add m xH yH middle in
let sum := combine n m sumL sumH in
 carry_add (n + m) (combine n m xL xH)(combine n m yL yH) cin = (sum,cout).
```

```
Instance ADD_Spec cin a b cout sum n : Implement
(ι_cin • (Φ^n_a • Φ^n_b))
(Φ^n_sum • ι_cout)
(ADD cin a b cout sum n)
(fun (c,(x,y)) ⇒ carry_add c x y).
```

Some sub-components



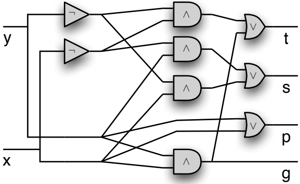
Definition HL x n p : $\mathbb{C}$ $((n+p)\cdot \mathbf{1}_{\mathrm{x}})$ $(n\cdot \mathbf{1}_{\mathrm{x}} \oplus p\cdot \mathbf{1}_{\mathrm{x}})$:= Plug ...
Definition COMBINE x n p : $\mathbb{C}$ $(n\cdot \mathbf{1}_{\mathrm{x}} \oplus p\cdot \mathbf{1}_{\mathrm{x}})$ $((n+p)\cdot \mathbf{1}_{\mathrm{x}})$:= Plug ...

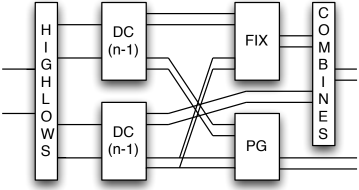
Instance HL_Spec x n p: Implement
$(\Phi_{\mathrm{x}}^{\mathrm{n+p}})$ $(\Phi_{\mathrm{x}}^{\mathrm{n}} \bullet \Phi_{\mathrm{x}}^{\mathrm{p}})$ (HL x n p) (fun x $\Rightarrow$ (low n p x, high n p x)).

Instance COMBINE_Spec x n p: Implement
$(\Phi_{\mathrm{x}}^{\mathrm{n}} \bullet \Phi_{\mathrm{x}}^{\mathrm{p}})$ $(\Phi_{\mathrm{x}}^{\mathrm{n+p}})$ (COMBINE x n p) (fun x $\Rightarrow$ (combine n p (fst x) (snd x))).

A divide and conquer adder

- Add in parallel the high-order and the low-order bits.
- Computes *s* (resp. *t*) the sum without (resp. with) a carry-in
- Computes *p* the carry-propagate and *g* the carry-generate





Implements the following Coq (high-level) function:

Definition dc n : $\mathbb{W}_{2^n} * \mathbb{W}_{2^n} \rightarrow \mathbb{B} * \mathbb{B} * \mathbb{W}_{2^n} * \mathbb{W}_{2^n}$:= fun (x,y) ⇒
let (s,g) := carry_add 2^n x y false in
let (t,p) := carry_add 2^n x y true in (g,p,s,t).

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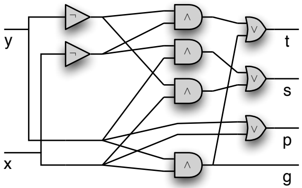

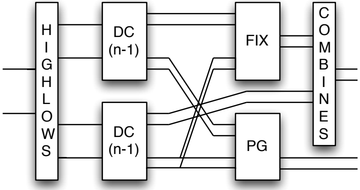

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```
Definition dc n : W_{2^n} * W_{2^n} → B * B * W_{2^n} * W_{2^n} := fun (x,y) ⇒
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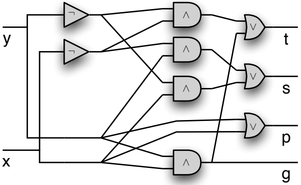
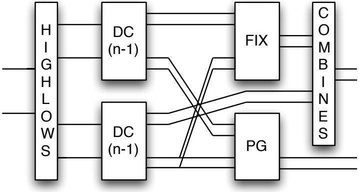
A divide and conquer adder





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- The high-level specification says nothing of the computationnal behavior of the circuit.
- The deep-embedding makes it possible to study the latency of this circuit.

Outline

Streams

In this section $\mathbb{T}$ is `stream` $\mathbb{B}$ i.e., `nat` $\rightarrow \mathbb{B}$.
We have several interesting isomorphisms:

```
Definition Iso_stream A B C (I: (A → B) ≅ C) : (A → stream B) ≅ (stream C) := ...
Definition Iso_prod_stream : (stream A * stream B) ≅ (stream (A * B)) := ...
Definition Iso_vector_stream n : (vector (stream A) n) ≅ (stream (vector A n)) := ...
```

Examples:

$(\mathbf{1} \oplus \mathbf{1} \rightarrow \texttt{nat} \rightarrow \mathbb{B}) \cong (\texttt{stream}\ (\mathbb{B} * \mathbb{B}))$
$(n \cdot \mathbf{1} \rightarrow \texttt{nat} \rightarrow \mathbb{B}) \cong (\texttt{stream}\ (\mathbb{W}_{\texttt{n}}))$

We assume (through appropriate parametrization) a gate `DFF` that implements a `pre`:

```
Definition pre {A} (d : A):
stream A → stream A := fun f t ⇒
match t with | 0 ⇒ d | S p ⇒ f p end.
```

```
Instance DFF_Realise_stream {a out}:
Implement (DFF a out) (ι_a) (ι_out)
(pre false).
```

Not in this talk: buffers defined with circuit combinators

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Lifting combinatorial circuits

Well-behaved circuits

"Circuits that have the same behavior in the boolean setting and pointwise in a stream"

A well-behaved atom satisfies:

$$\forall ins, \forall outs, \texttt{stream}\,\mathbb{B} \models ins \bowtie outs \implies \forall t, \mathbb{B} \models ins@t \bowtie outs@t.$$

Plugs are well-behaved, as well as parallel and serial circuits when their sub-circuits are well-behaved.

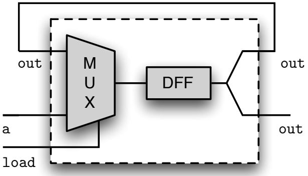
Lifting

```
Lemma lifting_map n m x (Hwf: wb n m x) N M
(Rn : (n → 𝔹) ≅ N)
(Rm : (m → 𝔹) ≅ M)
(f : N → M):
Implement 𝔹 ... Rn Rm x f →
Implement (stream 𝔹) ... (Iso_stream Rn) (Iso_stream Rm) x (Stream.map f).
```

A memory element

We can also deal with state-holding structures:

```
Definition REGISTER: ℂ (1_load ⊕ 1_a) 1_out :=
Loop (1_load ⊕ 1_a) 1_out 1_out
  (Plug ... ▷ MUX2 a out load "in_dff"
            ▷ DFF "in_dff" out
            ▷ Fork 2 1_out).
```



This circuit realise the following relation:

```
Record reg_ti := {va : bool; vload : bool}.
Instance Register_Spec : Realise (... : 1_load ⊕ 1_a → stream 𝔹 ≅ stream reg_ti) (ι_out) REGISTER
  (fun ins outs ⇒ outs = pre false (fun t ⇒ if (ins t).(vload) then (ins t).(va) else outs t)).
```

Note

Relations on streams are not the nicest way to reason about state-holding devices.

A memory element

We can also deal with state-holding structures:

Definition REGISTER: $\mathbb{C}$ $(\mathbf{1}_{load} \oplus \mathbf{1}_{a})$ $\mathbf{1}_{out}$:=
Loop $(\mathbf{1}_{load} \oplus \mathbf{1}_{a})$ $\mathbf{1}_{out}$ $\mathbf{1}_{out}$
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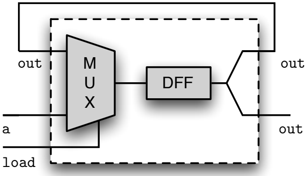

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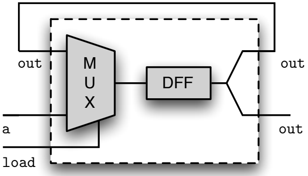

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```

Note

Relations on streams are not the nicest way to reason about state-holding devices.

Using Moore automata as specifications

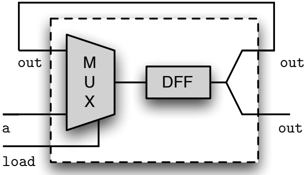
```
Record Moore I O σ := { λ : σ → O; δ : σ → I → σ}.

Fixpoint iterate (M: Moore I O σ) (x: σ) (ins: stream I) (k: nat): σ :=
 match k with | O ⇒ x | S n ⇒ M.(δ) (iterate M x ins n) (ins n) end.

Definition outputs (M: Moore I O σ) (x: σ) (ins: nat → I) k : O := M.(λ)  (iterate M x ins k).
```

A memory element (cont)

Definition REGISTER: $\mathbb{C}$ ($\mathbf{1}_{\mathtt{load}} \oplus \mathbf{1}_{\mathtt{a}}$) $\mathbf{1}_{\mathtt{out}}$:=
Loop ($\mathbf{1}_{\mathtt{load}} \oplus \mathbf{1}_{\mathtt{a}}$) $\mathbf{1}_{\mathtt{out}}$ $\mathbf{1}_{\mathtt{out}}$
  (Plug ... ▷ MUX2 a out load "in_dff"
    ▷ DFF "in_dff" out
    ▷ Fork 2 $\mathbf{1}_{\mathtt{out}}$).



Record reg_ti := {va : bool; vload : bool}.
Definition reg_m : Moore reg_ti bool bool :=
 {| λ := id; δ := fun state ins ⇒ if ins.(vload) then ins.(va) else state|}.

Instance Register_Spec : Realise (... : $\mathbf{1}_{\mathtt{load}} \oplus \mathbf{1}_{\mathtt{a}} \to$ stream $\mathbb{B} \cong$ stream reg_ti) ($\iota_{\mathtt{out}}$) REGISTER
 (fun ins outs ⇒ outs = Moore.outputs reg_m false ins).

Simulation and other stuff

Simulation:

- This first-order encoding makes it possible to simulate circuits inside Coq.
- It requires a computational interpretation of each basic gate.

```
Variable sem : ∀ a b, atom a b → (a → 𝕋) → option (b → 𝕋).
Fixpoint simulation n m (c : circuit atom n m) : (n → 𝕋) → option (m → 𝕋) := ...
```

- Not very efficient...

Using the same idea:

- computation of the gate-count and length of the critical path;
- pretty-printing of the list of gates (and their connection).

Almost to VHDL

Conclusion

- A deep-embedding of circuits in Coq
- Build and reason about circuits, proving high-level specifications through type-isomorphisms
- Dependent types are useful to capture some well-formedness properties
- Examples: arithmetic circuits of parametric size, proved by induction

Related works

- Verifying circuits with theorem provers:
  - in HOL or in Coq, mainly shallow-embeddings;
  - in HOL, a compiler from HOL to VHDL;
  - in ACL2, shallow-embeddings (up to a microprocessor), not higher-order.
  - in PVS, a shallow-embedding of a pipelined micro-processor
- Algebraic definitions of circuits:
  - Lafont: algebraic theory of boolean circuits.
  - Hinze: the algebra of parallel prefix circuits.
- Functional languages in hardware design:
  - Many approaches based on circuits combinators (often lack dependent types)
  - Lava: language embedded in Haskell to describe circuits (combinator based, but use names for wires)

- More arithmetic circuits.
- Transformations.
- Some front-end to generate circuits.

When I am PhDone:

```
Inductive aexp : nat → Type :=
| AVar: ∀ n (i : Var E n), aexp n
| AConst: ∀ n, Word.word n → aexp n
| APlus: ∀ n, aexp n → aexp n → aexp n
| ALo: ∀ n m, aexp (n + m) → aexp n
| AHi: ∀ n m, aexp (n + m) → aexp m
| ACat: ∀ n m, aexp n → aexp m → aexp (n + m).
```

```
Inductive bexp : Type :=
| BTrue: bexp
| BFalse: bexp
| BEq: ∀ n, aexp n → aexp n → bexp
| BLt: ∀ n, aexp n → aexp n → bexp
| BNot: bexp → bexp
| BAnd: bexp → bexp → bexp.
```

```
Inductive com (E: list nat) : Type :=
| CSkip: com E
| CAss: ∀ n (v : Var E n) , aexp E n → com E
| CSeq: com E → com E → com E
| CIf: bexp E → com E → com E → com E
| CWhile: bexp E → com E → com E
| CNew: ∀ n, aexp E n → com (snoc E n) → com E
```

- More arithmetic circuits.
- Transformations.
- Some front-end to generate circuits.

When I am PhDone:

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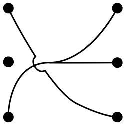
One more word on Plugs

- Thanks to the use of tags, plugs can be defined by proof search.
- ... but, for each plug, we have to exhibit the fonction it implements (up to isos).
- A better solution for simple plugs is to use the following definition.

```
Inductive monoid : Type :=
| Var : Type → monoid
| • : monoid → monoid → monoid.
```

```
Inductive ⊢ : monoid → monoid → Set :=
| M : ∀ A B C, A ⊢ C → B ⊢ C → (A • B) ⊢ C
| L : ∀ A B C, A ⊢ B → A ⊢ (B • C)
| R : ∀ A B C, A ⊢ B → A ⊢ (C • B)
| I : ∀ A, A ⊢ A.
```



$\mathbb{C}\ (n \oplus m \oplus p)\ (p \oplus (n \oplus n))$

$$\dfrac{\dfrac{I}{p \vdash (n \bullet m) \bullet p}\,R \qquad \dfrac{\dfrac{\dfrac{I \quad I}{n \bullet n \vdash n}\,M}{n \bullet n \vdash n \bullet m}\,L}{n \bullet n \vdash (n \bullet m) \bullet p}\,L}{p \bullet (n \bullet n) \vdash (n \bullet m) \bullet p}\,M$$

- Evaluated to $(p \oplus (n \oplus n)) \to (n \oplus m \oplus p)$ (the plug)
- Evaluated to $(\overline{n} \otimes \overline{m} \otimes \overline{p}) \to (\overline{p} \otimes (\overline{n} \otimes \overline{n}))$ (the action of the plug on values)

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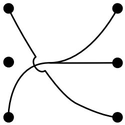

$\mathbb{C}\ (n \oplus m \oplus p)\ (p \oplus (n \oplus n))$

$$\mathrm{M}\ \dfrac{\mathrm{R}\ \dfrac{I}{p \vdash (n \bullet m) \bullet p} \qquad \mathrm{L}\ \dfrac{\mathrm{L}\ \dfrac{\mathrm{M}\ \dfrac{I \qquad I}{n \bullet n \vdash n}}{n \bullet n \vdash n \bullet m}}{n \bullet n \vdash (n \bullet m) \bullet p}}{p \bullet (n \bullet n) \vdash (n \bullet m) \bullet p}$$

- Evaluated to $(p \oplus (n \oplus n)) \to (n \oplus m \oplus p)$ (the plug)
- Evaluated to $(\overline{n} \otimes \overline{m} \otimes \overline{p}) \to (\overline{p} \otimes (\overline{n} \otimes \overline{n}))$ (the action of the plug on values)