A tutorial implementation
of a dependently typed lambda calculus

Andres Löh
Utrecht University
andres@cs.uu.nl

Conor McBride
University of Strathclyde
conor.mcbride@cis.strath.ac.uk

Wouter Swierstra
University of Nottingham
wss@cs.nott.ac.uk

Abstract. We present the type rules for a dependently typed core calculus together with a straightforward implementation in Haskell. We explicitly highlight the changes necessary to shift from a simply-typed lambda calculus to the dependently typed lambda calculus. We also describe how to extend our core language with data types and write several small example programs. The article is accompanied by an executable interpreter and example code that allows immediate experimentation with the system we describe.

1.   Introduction

Most functional programmers are hesitant to program with dependent types. It is said that type checking becomes undecidable; the type checker will always loop; and that dependent types are just really, really, hard.

The same programmers, however, are perfectly happy to program with a ghastly hodgepodge of complex type system extensions. Current Haskell implementations, for instance, support generalized algebraic data types, multi-parameter type classes with functional dependencies, associated types and type families, impredicative higher-ranked types, and there are even more extensions on the way. Programmers seem to be willing to go to great lengths just to avoid dependent types.

One of the major barriers preventing the further proliferation of dependent types is a lack of understanding amongst the general functional programming community. While, by now, there are quite a few good experimental tools and programming languages based on dependent types, it is hard to grasp how

these tools actually work. A significant part of the literature available on dependent types is written by type theorists for other type theorists to read. As a result, these papers are often not easily accessible to functional programmers. This article aims to remedy this situation.

Most importantly, we aim to fill a gap in the literature. To set the scene, we study the simply-typed lambda calculus (Section 2). We present both the mathematical specification and Haskell implementation of the abstract syntax, evaluation, and type checking. Taking the simply-typed lambda calculus as starting point, we move on to a minimal dependently typed lambda calculus (Section 3).

Inspired by Pierce's incremental development of type systems [21], we highlight the changes, both in the specification and implementation, that are necessary to shift to the dependently typed lambda calculus. Perhaps surprisingly, the modifications necessary are comparatively small. By making these changes as explicit as possible, we hope that the transition to dependent types will be as smooth as possible for readers already familiar with the simply-typed lambda calculus.

While none of the type systems we implement are new, we believe that our paper can serve as a gentle introduction on how to implement a dependently typed system in Haskell. Implementing a type system is one of the best ways to learn about all the subtle issues involved. Although we do not aim to survey all the different ways to implement a typed lambda calculus, we do try to be explicit about our design decisions, carefully mention alternative choices, and provide an outline of the wider design space.

The full power of dependent types can only come to its own if we add data types to this base calculus. Therefore we demonstrate how to extend our language with natural numbers and vectors in Section 4. More data types can be added using the principles explained in this section. Using the added data types, we write the classic vector append operation to illustrate how to program in our core calculus.

Finally, we have made it easy to experiment with our system: the source code of this article contains a small interpreter for the type system and evaluation rules we describe. By using the same sources as the article, the interpreter is guaranteed to follow the implementation we describe closely, and is carefully documented. It hence provides a valuable platform for further education and experimentation.

This article is *not* an introduction to dependently typed programming or an explanation on how to implement a full dependently typed programming language. However, we hope that this article will help to dispel many misconceptions functional programmers may have about dependent types, and that it will encourage readers to explore this exciting area of research further.

2.   Simply Typed Lambda Calculus

On our journey to dependent types, we want to start on familiar ground. In this section, we therefore consider the simply-typed lambda calculus, or $\lambda_\rightarrow$ for short. In a sense, $\lambda_\rightarrow$ is the smallest imaginable statically typed functional language. Every term is explicitly typed and no type inference is performed. It has a much simpler structure than the type lambda calculi at the basis of languages such as ML or Haskell that support polymorphic types and type constructors. In $\lambda_\rightarrow$, there are only base types and functions cannot be polymorphic. Without further additions, $\lambda_\rightarrow$ is strongly normalizing: evaluation terminates for any term, independent of the evaluation strategy.

2.1.   Abstract syntax

The type language of $\lambda_\rightarrow$ consists of just two constructs:

$$\frac{e \Downarrow v}{e :: \tau \Downarrow v} \quad \frac{}{x \Downarrow x}$$

$$\frac{e \Downarrow \lambda x \to v \quad v[x \mapsto e'] \Downarrow v'}{e\,e' \Downarrow v'} \quad \frac{e \Downarrow n \quad e' \Downarrow v'}{e\,e' \Downarrow n\,v'} \quad \frac{e \Downarrow v}{\lambda x \to e \Downarrow \lambda x \to v}$$

Figure 1.   Evaluation in $\lambda_\to$

$$
\begin{array}{lll}
\tau ::= \alpha & & \text{base type} \\
\quad | \quad \tau \to \tau' & & \text{function type}
\end{array}
$$

There is a set of base types α; compound types $\tau \to \tau'$ correspond to functions from τ to τ'.

$$
\begin{array}{lll}
e ::= e :: \tau & & \text{annotated term}[1] \\
\quad | \quad x & & \text{variable} \\
\quad | \quad e\,e' & & \text{application} \\
\quad | \quad \lambda x \to e & & \text{lambda abstraction}
\end{array}
$$

There are four kinds of terms: terms with an explicit type annotation; variables; applications; and lambda abstractions.

Terms can be evaluated to values:

$$
\begin{array}{lll}
v ::= n & & \text{neutral term} \\
\quad | \quad \lambda x \to v & & \text{lambda abstraction} \\
n ::= x & & \text{variable} \\
\quad | \quad n\,v & & \text{application}
\end{array}
$$

A value is either a *neutral term*, i.e., a variable applied to a (possibly empty) sequence of values, or it is a lambda abstraction.

2.2.   Evaluation

The (big-step) evaluation rules of $\lambda_\to$ are given in Figure 1. The notation $e \Downarrow v$ means that the result of completely evaluating e is v. Since we are in a strongly normalizing language, the evaluation strategy is irrelevant. To keep the presentation simple, we evaluate everything as far as possible, and even evaluate under lambda. Type annotations are ignored during evaluation. Variables evaluate to themselves.

The only interesting case is application. In that case, it depends whether the left subterm evaluates to a neutral term or a lambda abstraction. In the former case, the evaluation cannot proceed further and we construct a new neutral term from the results of evaluating the two subterms.

When evaluation of the left subterm does yield a lambda abstraction, we β-reduce. As this substitution may itself produce new redexes, we evaluate the result of the substitution.

Here are few example terms in $\lambda_\to$, and their evaluations. Let us write id to denote the term $\lambda x \to x$, and const to denote the term $\lambda x\,y \to x$, which we use in turn as syntactic sugar for $\lambda x \to \lambda y \to x$. Then

---

[1]Type theorists use ':' or '$\in$' to denote the type inhabitation relation. In Haskell, the symbol ':' is used as the "cons" operator for lists, therefore the designers of Haskell chose the non-standard '::' for type annotations. In this paper, we will stick as close as possible to Haskell's syntax, in order to reduce the syntactic gap between the languages involved in the presentation.

$$\Gamma ::= \varepsilon \qquad\qquad \text{empty context}$$
$$\mid\ \Gamma, \alpha :: * \qquad \text{adding a type identifier}$$
$$\mid\ \Gamma, x :: \tau \qquad \text{adding a term identifier}$$

$$\frac{}{\mathsf{valid}(\varepsilon)} \qquad \frac{\mathsf{valid}(\Gamma)}{\mathsf{valid}(\Gamma, \alpha :: *)} \qquad \frac{\mathsf{valid}(\Gamma) \quad \Gamma \vdash \tau :: *}{\mathsf{valid}(\Gamma, x :: \tau)}$$

$$\frac{\Gamma(\alpha) = *}{\Gamma \vdash \alpha :: *} \ (\text{TVAR}) \qquad \frac{\Gamma \vdash \tau :: * \quad \Gamma \vdash \tau' :: *}{\Gamma \vdash \tau \to \tau' :: *} \ (\text{FUN})$$

Figure 2.   Contexts and well-formed types in $\lambda_\to$

$$(\mathsf{id} :: \alpha \to \alpha)\ y \Downarrow y$$
$$(\mathsf{const} :: (\beta \to \beta) \to \alpha \to \beta \to \beta)\ \mathsf{id}\ y \Downarrow \mathsf{id}$$

2.3.   Type System

Type rules are generally of the form $\Gamma \vdash e :: t$, indicating that a term e is of type t in context Γ. The context lists valid base types, and associates identifiers with type information. We write $\alpha :: *$ to indicate that α is a base type, and $x :: t$ to indicate that x is a term of type t. Every free variable in both terms and types must occur in the context. For instance, if we want to declare const to be of type $(\beta \to \beta) \to \alpha \to \beta \to \beta$, we need our context to contain at least:

$$\alpha :: *, \beta :: *, \mathsf{const} :: (\beta \to \beta) \to \alpha \to \beta \to \beta$$

Note α and β are introduced before they are used in the type of const. These considerations motivate the definitions of contexts and their validity given in Figure 2.

Multiple bindings for the same variable can occur in a context, with the rightmost binding taking precedence. We write $\Gamma(z)$ to denote the information associated with identifier z by context Γ.

The last two rules in Figure 2 (TVAR, FUN) explain when a type is well-formed, i.e., when all its free variables appear in the context. In the rules for the well-formedness of types as well as in the type rules that follow, we implicitly assume that all contexts are valid.

Note that $\lambda_\to$ is not polymorphic: a type identifier represents one specific type and cannot be instantiated.

Finally, we can give the type rules (Figure 3). We do not try to infer the types of lambda-bound variables. Therefore, in general, we perform only type checking. However, for annotated terms, variables, and applications we can easily determine the type. We therefore mark type rules with $::_\downarrow$ when the type is supposed to be an input to the type checking algorithm and with $::_\uparrow$ when the type is produced by the type checking algorithm. For now, this is only to provide an intuition, but the distinction will become more significant in the implementation.

Let us first look at the inferable terms. We check annotated terms against their type annotation, and then return the type (ANN). The type of a variable can be looked up in the environment (VAR). For applications (APP), we deal with the function first, which must be of a function type. We can then check the argument against the function's domain, and return the range as the result type.

$$\frac{\Gamma \vdash \tau :: * \quad \Gamma \vdash e ::_\downarrow \tau}{\Gamma \vdash (e :: \tau) ::_\uparrow \tau} \text{ (ANN)} \qquad \frac{\Gamma(x) = \tau}{\Gamma \vdash x ::_\uparrow \tau} \text{ (var)} \qquad \frac{\Gamma \vdash e ::_\uparrow \tau \to \tau' \quad \Gamma \vdash e' ::_\downarrow \tau}{\Gamma \vdash e\, e' ::_\uparrow \tau'} \text{ (APP)}$$

$$\frac{\Gamma \vdash e ::_\uparrow \tau}{\Gamma \vdash e ::_\downarrow \tau} \text{ (CHK)} \qquad \frac{\Gamma, x :: \tau \vdash e ::_\downarrow \tau'}{\Gamma \vdash \lambda x \to e ::_\downarrow \tau \to \tau'} \text{ (LAM)}$$

Figure 3. Type rules for $\lambda_\to$

The final two rules are for type checking. If we can infer a type for a term, we can also check it against a type if the two types are identical (CHK). A lambda abstraction (LAM) can only be checked against a function type. We check the body of the abstraction in an extended context.

Note that the rules are almost syntax-directed: The rule relating checkable and inferable terms (CHK) seems to match any term. However, since there is no rule to infer the type of a lambda abstraction and there are no explicit rules to check an annotation, variable or application, the rules can easily be translated into a syntax-directed algorithm.

Here are type judgements – derivable using the above rules – for our two running examples:

$$\alpha :: *, y :: \alpha \qquad \vdash (\mathsf{id} :: \alpha \to \alpha)\, y :: \alpha$$
$$\alpha :: *, y :: \alpha, \beta :: * \vdash (\mathsf{const} :: (\beta \to \beta) \to \alpha \to \beta \to \beta)\ \mathsf{id}\ y :: \beta \to \beta$$

2.4. Implementation

We now give an implementation of $\lambda_\to$ in Haskell. We provide an evaluator for well-typed terms, and functions to type-check $\lambda_\to$ terms. The implementation follows the formal description that we have just introduced very closely.

There is a certain freedom in how to implement the rules. We pick an implementation that allows us to follow the type system closely, and that reduces the amount of technical overhead to a relative minimum, so that we can concentrate on the essence of the algorithms involved. In what follows, we briefly discuss our design decisions and mention alternatives. It is important to point out that none of these decisions is essential for implementing dependent types.

Representing bound variables There are different possibilities to represent bound variables – all of them have advantages, and in order to exploit a maximum of advantages, we choose different representations in different places of our implementation.

We represent locally bound variables by *de Bruijn indices*: variable occurrences are represented by numbers instead of strings or letters, the number indicating how many binders occur between its binder and the occurrence. For example, we can write id as $\lambda \to 0$, and const as $\lambda \to \lambda \to 1$ using de Bruijn indices. The advantage of this representation is that variables never have to be renamed, i.e., α-equality of terms reduces to syntactic equality of terms.

The disadvantage of using de Bruijn indices is it is rather cumbersome to manipulate terms with free variables. Although we can use indices unbound by a lambda to indicate that a variable is free, these

indices are *relative* – whenever we traverse a term and go under a lambda, these references must be updated accordingly.

We therefore represent such free variables in terms using *absolute* references, i.e., *names*. The combination of using numbers for variables local, and names for variables global to the current term is called a *locally nameless* representation [23, 13].

Finally, we use *higher-order abstract syntax* to represent values: values that are functions are represented using Haskell functions. This has the advantage that we can use Haskell's function application and do not have to implement substitution ourselves, and need not worry about name capture. A slight downside of this approach is that Haskell functions can neither be shown nor compared for equality. Fortunately, this drawback can easily be alleviated by *quoting* a value back into a concrete representation. We will return to quoting once we have defined the evaluator and the type checker.

Separating inferable and checkable terms   As we have already hinted at in the presentation of the type rules for $\lambda_\rightarrow$ in Figure 3, we choose to distinguish terms for which the type can be read off (called *inferable terms*) and terms for which we need a type to check them. This syntactic distinction between checkable and inferable terms dates at least as far back as work by Pierce and Turner [22].

Alternatively, we could require every lambda-abstracted variable to be explicitly annotated in the abstract syntax – we would then have inferable terms exclusively. It is, however, very useful to be able to annotate any term. In the presence of general annotations, it is no longer necessary to require an annotation on every lambda-bound variable. In fact, allowing un-annotated lambdas gives us quite a bit of convenience without extra cost: applications of the form $e\ (\lambda x \rightarrow e')$ can be processed without type annotation, because the type of x is determined by the type of e.

Abstract syntax   We introduce data types for *inferable* ($\mathsf{Term}_\uparrow$) and *checkable* ($\mathsf{Term}_\downarrow$) terms, and for names.

```
data Term↑
   = Ann    Term↓ Type
   | Bound Int
   | Free   Name
   | Term↑ :@: Term↓
   deriving (Show, Eq)
data Term↓
   = Inf   Term↑
   | Lam Term↓
   deriving (Show, Eq)
data Name
   = Global String
   | Local   Int
   | Quote   Int
   deriving (Show, Eq)
```

Annotated terms are represented using *Ann*. As explained above, we use integers to represent bound variables (*Bound*), and names for free variables (*Free*). Names usually refer to global entities using strings.

When passing a binder in an algorithm, we have to convert a bound variable into a free variable temporarily, and use *Local* for that. During quoting, we will use the *Quote* constructor. The infix constructor :@: denotes application.

Inferable terms are embedded in the checkable terms via the constructor *Inf*, and lambda abstractions (which do not introduce an explicit variable due to our use of de Bruijn indices) are written using *Lam*.

Types consist only of type identifiers (*TFree*) or function arrows (*Fun*). We reuse the Name data type for type identifiers. In $\lambda_\rightarrow$, there are no bound names on the type level, so there is no need for a *TBound* constructor.

```
data Type
  = TFree Name
  | Fun   Type Type
  deriving (Show, Eq)
```

Values are lambda abstractions (*VLam*) or neutral terms (*VNeutral*).

```
data Value
  = VLam    (Value → Value)
  | VNeutral Neutral
```

As described in the discussion on higher-order abstract syntax, we represent function values as Haskell functions of type Value → Value. For instance, the term const – when evaluated – results in the value *VLam* ($\lambda x \rightarrow$ *VLam* ($\lambda y \rightarrow x$)).

The data type for neutral terms matches the formal abstract syntax exactly. A neutral term is either a variable (*NFree*), or an application of a neutral term to a value (*NApp*).

```
data Neutral
  = NFree Name
  | NApp  Neutral Value
```

We introduce a function *vfree* that creates the value corresponding to a free variable:

```
vfree :: Name → Value
vfree n = VNeutral (NFree n)
```

Evaluation   The code for evaluation is given in Figure 4. The functions $eval_\uparrow$ and $eval_\downarrow$ implement the big-step evaluation rules for inferable and checkable terms respectively. Comparing the code to the rules in Figure 1 reveals that the implementation is mostly straightforward.

Substitution is handled by passing around an environment of values. Since bound variables are represented as integers, the environment is just a list of values where the i-th position corresponds to the value of variable i. We add a new element to the environment whenever evaluating underneath a binder, and lookup the correct element (using Haskell's list lookup operator (!!)) when we encounter a bound variable.

For lambda functions (*Lam*), we introduce a Haskell function and add the bound variable x to the environment while evaluating the body.

```
type Env = [Value]

eval↑ :: Term↑ → Env → Value
eval↑ (Ann e _) d = eval↓ e d
eval↑ (Free x)  d = vfree x
eval↑ (Bound i) d = d !! i
eval↑ (e :@: e') d = vapp (eval↑ e d) (eval↓ e' d)

vapp :: Value → Value → Value
vapp (VLam f)    v = f v
vapp (VNeutral n) v = VNeutral (NApp n v)

eval↓ :: Term↓ → Env → Value
eval↓ (Inf  i) d = eval↑ i d
eval↓ (Lam e) d = VLam (λx → eval↓ e (x : d))
```

Figure 4.   Implementation of an evaluator for $\lambda_\rightarrow$

Contexts   Before we can tackle the implementation of type checking, we have to define contexts. Contexts are implemented as (reversed) lists associating names with either ∗ (*HasKind Star*) or a type (*HasType t*):

```
data Kind = Star
    deriving (Show)
data Info
    = HasKind Kind
    | HasType Type
    deriving (Show)
type Context = [(Name, Info)]
```

Extending a context is thus achieved by the list "cons" operation; looking up a name in a context is performed by the Haskell standard list function *lookup*.

Type checking   We now implement the rules in Figure 3. The code is shown in Figure 5. The type checking algorithm can fail, and to do so gracefully, it returns a result in the Result monad. For simplicity, we choose a standard error monad in this presentation:

```
type Result α = Either String α
```

We use the function *throwError* :: String → Result α to report an error.

The function for inferable terms $type_\uparrow$ returns a type, whereas the function for checkable terms $type_\downarrow$ takes a type as input and returns (). The well-formedness of types is checked using the function $kind_\downarrow$. Each case of the definitions corresponds directly to one of the rules.

The type-checking functions are parameterized by an integer argument indicating the number of binders we have encountered. On the initial call, this argument is 0, therefore we provide $type_{\uparrow 0}$ as a wrapper function.

$kind_\downarrow$ :: Context $\rightarrow$ Type $\rightarrow$ Kind $\rightarrow$ Result ()
$kind_\downarrow \ \Gamma \ (TFree \ x) \ Star$
   = **case** *lookup* $x \ \Gamma$ **of**
       Just (*HasKind Star*) $\rightarrow$ *return* ()
       Nothing $\rightarrow$ *throwError* `"unknown identifier"`
$kind_\downarrow \ \Gamma \ (Fun \ \kappa \ \kappa') \ Star$
   = **do** $kind_\downarrow \ \Gamma \ \kappa \ \ Star$
       $kind_\downarrow \ \Gamma \ \kappa' \ Star$

$type_{\uparrow 0}$ :: Context $\rightarrow$ Term$_\uparrow$ $\rightarrow$ Result Type
$type_{\uparrow 0} = type_\uparrow \ 0$
$type_\uparrow$ :: Int $\rightarrow$ Context $\rightarrow$ Term$_\uparrow$ $\rightarrow$ Result Type
$type_\uparrow \ i \ \Gamma \ (Ann \ e \ \tau)$
   = **do** $kind_\downarrow \ \Gamma \ \tau \ Star$
       $type_\downarrow \ i \ \Gamma \ e \ \tau$
       return τ
$type_\uparrow \ i \ \Gamma \ (Free \ x)$
   = **case** *lookup* $x \ \Gamma$ **of**
       Just (*HasType* τ) $\rightarrow$ *return* τ
       Nothing $\rightarrow$ *throwError* `"unknown identifier"`
$type_\uparrow \ i \ \Gamma \ (e \ :@: e')$
   = **do** $\sigma \ \leftarrow type_\uparrow \ i \ \Gamma \ e$
      case σ **of**
        Fun $\tau \ \tau' \rightarrow$ **do** $type_\downarrow \ i \ \Gamma \ e' \ \tau$
                      return τ'
        _ $\rightarrow$ *throwError* `"illegal application"`
$type_\downarrow$ :: Int $\rightarrow$ Context $\rightarrow$ Term$_\downarrow$ $\rightarrow$ Type $\rightarrow$ Result ()
$type_\downarrow \ i \ \Gamma \ (Inf \ e) \ \tau$
   = **do** $\tau' \ \leftarrow type_\uparrow \ i \ \Gamma \ e$
      unless $(\tau \ \text{==} \ \tau')$ (*throwError* `"type mismatch"`)
$type_\downarrow \ i \ \Gamma \ (Lam \ e) \ (Fun \ \tau \ \tau')$
   = $type_\downarrow \ (i + 1) \ ((Local \ i, HasType \ \tau) : \Gamma)$
        $(subst_\downarrow \ 0 \ (Free \ (Local \ i)) \ e) \ \tau'$
$type_\downarrow \ i \ \Gamma \ \_ \ \_$
   = *throwError* `"type mismatch"`

Figure 5.   Implementation of a type checker for $\lambda_\rightarrow$

$subst_\uparrow :: \mathsf{Int} \to \mathsf{Term}_\uparrow \to \mathsf{Term}_\uparrow \to \mathsf{Term}_\uparrow$
$subst_\uparrow\ i\ r\ (Ann\ e\ \tau) = Ann\ (subst_\downarrow\ i\ r\ e)\ \tau$
$subst_\uparrow\ i\ r\ (Bound\ j) = \mathbf{if}\ i == j\ \mathbf{then}\ r\ \mathbf{else}\ Bound\ j$
$subst_\uparrow\ i\ r\ (Free\ y)\quad = Free\ y$
$subst_\uparrow\ i\ r\ (e :@: e') = subst_\uparrow\ i\ r\ e :@: subst_\downarrow\ i\ r\ e'$

$subst_\downarrow :: \mathsf{Int} \to \mathsf{Term}_\uparrow \to \mathsf{Term}_\downarrow \to \mathsf{Term}_\downarrow$
$subst_\downarrow\ i\ r\ (Inf\ e)\quad = Inf\ (subst_\uparrow\ i\ r\ e)$
$subst_\downarrow\ i\ r\ (Lam\ e)\quad = Lam\ (subst_\downarrow\ (i+1)\ r\ e)$

Figure 6.   Implementation of substitution for $\lambda_\to$

We use this integer to simulate the type rules in the handling of bound variables. In the type rule for lambda abstraction, we add the bound variable to the context while checking the body. We do the same in the implementation. The counter i indicates the number of binders we have passed, so *Local i* is a fresh name that we can associate with the bound variable. We then add *Local i* to the context Γ when checking the body. However, because we are turning a bound variable into a free variable, we have to perform the corresponding substitution on the body. The type checker will never encounter a bound variable; correspondingly the function $type_\uparrow$ has no case for *Bound*.

Note that the type equality check that is performed when checking an inferable term is implemented by a straightforward syntactic equality on the data type Type. Our type checker does not perform unification.

The code for substitution is shown in Figure 6, and again comprises a function for checkable ($subst_\downarrow$) and one for inferable terms ($subst_\uparrow$). The integer argument indicates which variable is to be substituted. The interesting cases are the one for *Bound* where we check if the variable encountered is the one to be substituted or not, and the case for *Lam*, where we increase i to reflect that the variable to substitute is referenced by a higher number underneath the binder.

Our implementation of the simply-typed lambda calculus is now almost complete. A small problem that remains is the evaluator returns a Value, and we currently have no way to print elements of type Value.

Quotation   As we mentioned earlier, the use of higher-order abstract syntax requires us to define a *quote* function that takes a Value back to a term. As the *VLam* constructor of the Value data type takes a function as argument, we cannot simply derive *Show* and *Eq* as we did for the other types. Therefore, as soon as we want to get back at the internal structure of a value, for instance to display results of evaluation, we need the function *quote*. The code is given in Figure 7.

The function *quote* takes an integer argument that counts the number of binders we have traversed. Initially, *quote* is always called with 0, so we wrap this call in the function $quote_0$.

If the value is a lambda abstraction, we generate a fresh variable *Quote i* and apply the Haskell function f to this fresh variable. The value resulting from the function application is then quoted at level $i + 1$. We use the constructor *Quote* that takes an argument of type Int here to ensure that the newly created names do not clash with other names in the value.

If the value is a neutral term (hence an application of a free variable to other values), the function

$quote_0 :: \mathsf{Value} \to \mathsf{Term}_\downarrow$
$quote_0 = quote\ 0$

$quote :: \mathsf{Int} \to \mathsf{Value} \to \mathsf{Term}_\downarrow$
$quote\ i\ (VLam\ f)\quad = Lam\ (quote\ (i+1)\ (f\ (vfree\ (Quote\ i))))$
$quote\ i\ (VNeutral\ n) = Inf\ (neutralQuote\ i\ n)$

$neutralQuote :: \mathsf{Int} \to \mathsf{Neutral} \to \mathsf{Term}_\uparrow$
$neutralQuote\ i\ (NFree\ x)\quad = boundfree\ i\ x$
$neutralQuote\ i\ (NApp\ n\ v) = neutralQuote\ i\ n\ :@:\ quote\ i\ v$

Figure 7.   Quotation in $\lambda_\to$

neutralQuote is used to quote the arguments. The *boundfree* function checks if the variable occurring at the head of the application is a *Quote* and thus a bound variable, or a free name:

$boundfree :: \mathsf{Int} \to \mathsf{Name} \to \mathsf{Term}_\uparrow$
$boundfree\ i\ (Quote\ k) = Bound\ (i-k-1)$
$boundfree\ i\ x\qquad\quad = Free\ x$

Quotation of functions is best understood by example. The value corresponding to the term const is $VLam\ (\lambda x \to VLam\ (\lambda y \to x))$. Applying $quote_0$ yields the following:

$$
\begin{aligned}
&\quad quote\ 0\ (VLam\ (\lambda x \to VLam\ (\lambda y \to x)))\\
&= Lam\ (quote\ 1\ (VLam\ (\lambda y \to vfree\ (Quote\ 0))))\\
&= Lam\ (Lam\ (quote\ 2\ (vfree\ (Quote\ 0))))\\
&= Lam\ (Lam\ (neutralQuote\ 2\ (NFree\ (Quote\ 0))))\\
&= Lam\ (Lam\ (Bound\ 1))
\end{aligned}
$$

When *quote* moves underneath a binder, we introduce a temporary name for the bound variable. To ensure that names invented during quotation do not interfere with any other names, we only use the constructor *Quote* during the quotation process. If the bound variable actually occurs in the body of the function, we will sooner or later arrive at those occurrences. We can then generate the correct de Bruijn index by determining the number of binders we have passed between introducing and observing the *Quote* variable.

Examples   We can now test the implementation on our running examples. We make the following definitions

$$
\begin{aligned}
&id' \quad = \mathit{Lam}\ (\mathit{Inf}\ (\mathit{Bound}\ 0)) \\
&const' = \mathit{Lam}\ (\mathit{Lam}\ (\mathit{Inf}\ (\mathit{Bound}\ 1))) \\[4pt]
&tfree\ \alpha = \mathit{TFree}\ (\mathit{Global}\ \alpha) \\
&free\ x\ \ = \mathit{Inf}\ (\mathit{Free}\ (\mathit{Global}\ x)) \\[4pt]
&term_1 = \mathit{Ann}\ id'\ (\mathit{Fun}\ (tfree\ \texttt{"a"})\ (tfree\ \texttt{"a"}))\ \mathbin{:@:} free\ \texttt{"y"} \\
&term_2 = \mathit{Ann}\ const'\ (\mathit{Fun}\ (\mathit{Fun}\ (tfree\ \texttt{"b"})\ (tfree\ \texttt{"b"})) \\
&\hspace{7.5em} (\mathit{Fun}\ (tfree\ \texttt{"a"}) \\
&\hspace{9.5em} (\mathit{Fun}\ (tfree\ \texttt{"b"})\ (tfree\ \texttt{"b"})))) \\
&\hspace{3em} \mathbin{:@:} id'\ \mathbin{:@:} free\ \texttt{"y"} \\[4pt]
&env_1\ \ = [\,(\mathit{Global}\ \texttt{"y"}, \mathit{HasType}\ (tfree\ \texttt{"a"})), \\
&\hspace{4em} (\mathit{Global}\ \texttt{"a"}, \mathit{HasKind}\ \mathit{Star})\,] \\
&env_2\ \ = [\,(\mathit{Global}\ \texttt{"b"}, \mathit{HasKind}\ \mathit{Star})\,] \mathbin{+\!\!+} env_1
\end{aligned}
$$

and then run an interactive session in Hugs or GHCi:[2]

$$
\begin{aligned}
&\rangle\ quote_0\ (eval_\uparrow\ term_1\ [\,]) \\
&\mathit{Inf}\ (\mathit{Free}\ (\mathit{Global}\ \texttt{"y"})) \\[4pt]
&\rangle\ quote_0\ (eval_\uparrow\ term_2\ [\,]) \\
&\mathit{Lam}\ (\mathit{Inf}\ (\mathit{Bound}\ 0)) \\[4pt]
&\rangle\ type_{\uparrow 0}\ env_1\ term_1 \\
&\mathit{Right}\ (\mathit{TFree}\ (\mathit{Global}\ \texttt{"a"})) \\[4pt]
&\rangle\ type_{\uparrow 0}\ env_2\ term_2 \\
&\mathit{Right}\ (\mathit{Fun}\ (\mathit{TFree}\ (\mathit{Global}\ \texttt{"b"}))\ (\mathit{TFree}\ (\mathit{Global}\ \texttt{"b"})))
\end{aligned}
$$

We have implemented a parser, pretty-printer and a small read-eval-print loop,[3] so that the above interaction can more conveniently take place as:

$$
\begin{aligned}
&\rangle\!\rangle\ \textbf{assume}\ (\alpha :: *)\ (y :: \alpha) \\[4pt]
&\rangle\!\rangle\ ((\lambda x \to x) :: \alpha \to \alpha)\ y \\
&y :: \alpha \\[4pt]
&\rangle\!\rangle\ \textbf{assume}\ \beta :: * \\[4pt]
&\rangle\!\rangle\ ((\lambda x\ y \to x) :: (\beta \to \beta) \to \alpha \to \beta \to \beta)\ (\lambda x \to x)\ y \\
&\lambda x \to x :: \beta \to \beta
\end{aligned}
$$

With **assume**, names are introduced and added to the context. For each term, the interpreter performs type checking and evaluation, and shows the final value and the type.

3.   Dependent types

In this section, we will modify the type system of the simply-typed lambda calculus into a dependently typed lambda calculus, called λ_Π. In the beginning of this section, we discuss the two core ideas of the

---

[2]Using lhs2TEX [6], one can generate a valid Haskell program from the sources of this paper. The results given here automatically generated by invoking GHCi whenever this paper is typeset.

[3]The code is included in the paper sources, but omitted from the typeset version for brevity.

upcoming changes. We then repeat the formal definitions of abstract syntax, evaluation and type rules, and highlight the changes with respect to the simply-typed case. We conclude this section by discussing how to adapt the implementation.

Dependent function space   In languages such as Haskell we can define *polymorphic* functions, such as the identity function:

$id :: \forall \alpha.\alpha \to \alpha$
$id = \lambda x \to x$

By using polymorphism, we can avoid writing the same function on, say, integers and booleans. Polymorphism can be made explicit by interpreting it as a type abstraction. The identity function then takes *two* arguments: a type α and a value of α. Calls to the new identity function must explicitly instantiate the identity function with a *type*:

$id :: \forall \alpha.\alpha \to \alpha$
$id = \lambda (\alpha :: *) \ (x :: \alpha) \to x$
id Bool *True* :: Bool
id Int 3 :: Int

Polymorphism thus allows types to abstract over types. Why would you want to do anything different? Consider the following data types:

data Vec0 $\alpha = Vec0$
data Vec1 $\alpha = Vec1 \ \alpha$
data Vec2 $\alpha = Vec2 \ \alpha \ \alpha$
data Vec3 $\alpha = Vec3 \ \alpha \ \alpha \ \alpha$

Clearly, there is a pattern here. We would like to have a single *family* of types, indexed by the number of elements,

$\forall \alpha :: *.\forall n ::$ Nat.Vec $\alpha \ n$

but we cannot easily do this in Haskell. The problem is that the *type* Vec abstracts over the *value n*.

The *dependent function space* '$\forall$' generalizes the usual function space '$\to$' by allowing the range to *depend* on the domain. The parametric polymorphism known from Haskell can be seen as a special case of a dependent function, motivating our use of the symbol '$\forall$'.[4] In contrast to polymorphism, the dependent function space can abstract over more than just types. The Vec type above is a valid dependent type.

It is important to note that the dependent function space is a generalization of the usual function space. We can, for instance, type the identity function on vectors as follows:

$\forall \alpha :: *.\forall n ::$ Nat.$\forall v ::$ Vec $\alpha \ n$.Vec $\alpha \ n$

---

[4]Type theorists call dependent function types Π-types and would write $\Pi \alpha : *.\Pi n :$ Nat.Vec $\alpha \ n$ instead. This is also why we call the calculus λ_Π.

Note that the type v does not occur in the range: this is simply the non-dependent function space already familiar to Haskell programmers. Rather than introduce unnecessary variables, such as v, we use the ordinary function arrow for the non-dependent case. The identity on vectors then has the following, equivalent, type:

$$\forall \alpha :: *.\forall n :: \mathsf{Nat}.\mathsf{Vec}\ \alpha\ n \to \mathsf{Vec}\ \alpha\ n$$

In Haskell, one can sometimes 'fake' the dependent function space [11], for instance by defining natural numbers on the type level (i.e., by defining data types Zero and Succ). Since the type-level numbers are different from the value level natural numbers, one then ends up duplicating a lot of concepts on both levels. Furthermore, even though one can lift certain values to the type level in this fashion, additional effort – in the form of advanced type class programming – is required to perform any computation on such types. Using dependent types, we can parameterize our types by *values*, and as we will shortly see, the normal evaluation rules apply.

Everything is a term   Allowing values to appear freely in types breaks the separation of terms, types, and kinds we mentioned in the introduction. There is no longer a syntactic distinction between these different levels: everything is a term. In Haskell, the symbol '::' relates entities on different syntactic levels: In 0 :: Nat, the 0 is syntactically a value and Nat a type, in Nat :: *, the Nat is a type and * is a kind. Now, *, Nat and 0 are all terms. While 0 :: Nat and Nat :: * still hold, the symbol '::' relates two terms. We will still use the word "type" to refer to terms ρ with $\rho :: *$, and still call * a "kind", but all these entities now reside on the same syntactic level. As a consequence, all language constructs are now available everywhere. In particular, we have abstraction and application of types and kinds.

We have now familiarized ourselves with the core ideas of dependently typed systems. Next, we discuss what we have to change in $\lambda_\to$ in order to accomplish these ideas and arrive at λ_Π.

3.1. Abstract syntax

We no longer have the need for a separate syntactic category of types or kinds, all constructs for all levels are now integrated into the term language:

$$
\begin{array}{llll}
e, \boxed{\rho, \kappa} ::= & e :: \rho & \text{annotated term} \\
| & * & \text{the type of types} \\
| & \boxed{\forall x :: \rho.\rho'} & \text{dependent function space} \\
| & x & \text{variable} \\
| & e\ e' & \text{application} \\
| & \lambda x \to e & \text{lambda abstraction}
\end{array}
$$

The modifications compared to the abstract syntax of the simply-typed lambda calculus in Section 2.1 are highlighted.

We now also use the symbols ρ and κ to refer to terms that play the role of types or kinds, respectively.

New constructs are imported from what was formerly in the syntax of types and kinds. The kind * is now an term. Arrow kinds and arrow types are subsumed by the new construct for the dependent function space. Type variables and term variables now coincide.

$$\frac{e \Downarrow v}{e :: \rho \Downarrow v} \qquad \frac{}{* \Downarrow *} \qquad \frac{\rho \Downarrow \tau \quad \rho' \Downarrow \tau'}{\forall x :: \rho.\rho' \Downarrow \forall x :: \tau.\tau'} \qquad \frac{}{x \Downarrow x}$$

$$\frac{e \Downarrow \lambda x \to v \quad v[x \mapsto e'] \Downarrow v'}{e\ e' \Downarrow v'} \qquad \frac{e \Downarrow n \quad e' \Downarrow v'}{e\ e' \Downarrow n\ v'} \qquad \frac{e \Downarrow v}{\lambda x \to e \Downarrow \lambda x \to v}$$

Figure 8.   Evaluation in λ_Π

$$
\begin{aligned}
\Gamma ::=\ & \varepsilon && \text{empty context} \\
\mid\ & \Gamma, x :: \tau && \text{adding a variable}
\end{aligned}
$$

$$\frac{}{\mathsf{valid}(\varepsilon)} \qquad \frac{\mathsf{valid}(\Gamma) \quad \Gamma \vdash \tau ::_\downarrow *}{\mathsf{valid}(\Gamma, x :: \tau)}$$

Figure 9.   Contexts in λ_Π

3.2.   Evaluation

The modified evaluation rules are in Figure 8. All the rules are the same as in the simply-typed case in Figure 1, only the rules for the two new constructs are added. Perhaps surprisingly, evaluation now also extends to types. But this is exactly what we want: the power of dependent types stems exactly from the fact that we can mix values and types, and that we have functions, and thus computations on the type level. However, the new constructs are comparatively uninteresting for computation: the $*$ evaluates to itself; in a dependent function space, we recurse structurally, evaluating the domain and the range. We must extend the abstract syntax of values accordingly:

$$
\begin{aligned}
v, \tau ::=\ & n && \text{neutral term} \\
\mid\ & * && \text{the type of types} \\
\mid\ & \forall x :: \tau.\tau' && \text{dependent function space} \\
\mid\ & \lambda x \to v && \text{lambda abstraction}
\end{aligned}
$$

We now use the symbol τ for values that play the role of types.

3.3.   Type system

Before we approach the type rules themselves, we must take a look at contexts again. It turns out that because everything is a term now, the syntax of contexts becomes simpler, as do the rules for the validity of contexts (Figure 9, compare with Figure 2).

 Contexts now contain only one form of entry, stating the type a variable is assumed to have. Note that we always store evaluated types in a context. The precondition $\Gamma \vdash \tau ::_\downarrow *$ in the validity rule for non-empty contexts no longer refers to a special judgement for the well-formedness of types, but to the type rules we are about to define – we no longer need special well-formedness rules for types. The precondition ensures in particular that τ does not contain unknown variables.

 The type rules are given in Figure 10. Type rules now relate a context, a term and a value, i.e., all types are evaluated as soon as possible. Again, we have highlighted the differences from Figure 3. We

$$\frac{\Gamma \vdash \rho ::_\downarrow * \quad \rho \Downarrow \tau \quad \Gamma \vdash e ::_\downarrow \tau}{\Gamma \vdash (e :: \rho) ::_\uparrow \tau} \ (\text{ANN})$$

$$\frac{}{\Gamma \vdash * ::_\uparrow *} \ (\text{STAR})$$

$$\frac{\Gamma \vdash \rho ::_\downarrow * \quad \rho \Downarrow \tau \quad \Gamma, x :: \tau \vdash \rho' ::_\downarrow *}{\Gamma \vdash \forall x :: \rho.\rho' ::_\uparrow *} \ (\text{PI})$$

$$\frac{\Gamma(x) = \tau}{\Gamma \vdash x ::_\uparrow \tau} \ (\text{VAR})$$

$$\frac{\Gamma \vdash e ::_\uparrow \forall x :: \tau.\tau' \quad \Gamma \vdash e' ::_\downarrow \tau \quad \tau'[x \mapsto e'] \Downarrow \tau''}{\Gamma \vdash e\ e' ::_\uparrow \tau''} \ (\text{APP})$$

$$\frac{\Gamma \vdash e ::_\uparrow \tau}{\Gamma \vdash e ::_\downarrow \tau} \ (\text{CHK})$$

$$\frac{\Gamma, x :: \tau \vdash e ::_\downarrow \tau'}{\Gamma \vdash \lambda x \to e ::_\downarrow \forall x :: \tau.\tau'} \ (\text{LAM})$$

Figure 10.　Type rules for λ_Π

maintain the difference between rules for inference ($::_\uparrow$), where the type is an output, and checking ($::_\downarrow$), where the type is an input. The new constructs $*$ and $\forall$ are among the constructs for which we can infer the type. As before for $\lambda_\to$, we assume that all the contexts that occur in the type rules are valid.

For annotated terms (ANN), there are two changes. The kind check for the annotation ρ no longer refers to the well-formedness rules for types, but is to the rules for type checking. Furthermore, because the annotation ρ might not be a value, it is evaluated before it is returned.

The kind $*$ is itself of type $*$ (STAR). Although there are theoretical objections to this choice (see Section 5), we believe that for the purpose of this paper, the simplicity of our implementation outweighs any such objections.

The rule for the dependent function space (PI) is somewhat similar to the well-formedness rule for arrow types (FUN) for $\lambda_\to$ in Figure 2. Both the domain ρ and the range ρ' of the dependent function are required to be of kind $*$. In contrast to the old rule FUN, ρ' may refer to x, thus we extend Γ by $x :: \tau$ (where τ is the result of evaluating τ') while checking ρ'.

In a function application (APP), the function must now be of a dependent function type $\forall x :: \tau.\tau'$. In contrast to an ordinary function type, τ' can refer to x. In the result type of the application, we therefore substitute the actual argument e' for the formal parameter x in τ'.

Checking an inferable term (CHK) works as before: given a term e and an type τ, we first infer a type for e, and then check that the inferred type is indeed equal to the expected type τ. However, we are now dealing with evaluated types, so this is much stronger than syntactic equality of type terms: it would be rather unfortunate if Vec α 2 and Vec α $(1 + 1)$ did not denote the same type. Our system indeed recognises them as equal, because both terms evaluate to Vec α 2. Most type systems with dependent types have a rule of the form:

$$\frac{\Gamma \vdash e :: \rho \quad \rho =_\beta \rho'}{\Gamma \vdash e :: \rho'}$$

This rule, referred to as the *conversion rule*, however, is clearly not syntax directed. The distinction between inferable and checkable terms ensures that the only place where we need to apply the conversion rule is when a term is explicitly annotated with its type (ANN).

The final type rule is for checking a lambda abstraction (LAM). The difference here is that the type is a dependent function type. Note that the bound variable x may now not only occur in the body of the function e. The extended context $\Gamma, x :: \tau$ is therefore used both for type checking the function body and kind checking the range τ'.

To summarize, all the modifications are motivated by the two key concepts we have introduced in the beginning of Section 3: the function space is generalized to the dependent function space; types and kinds are also terms.

3.4. Implementation

The type rules we have given are still syntax-directed and algorithmic, so the general structure of the $\lambda_\rightarrow$ implementation can be reused for λ_Π. In the following, we go through all aspects of the implementation, but only discuss the definitions that have to be modified.

Abstract syntax   We no longer require Type and Kind, but instead add two new constructors to $\text{Term}_\uparrow$ and replace the occurrence of Type in *Ann* with a $\text{Term}_\downarrow$:

```
data Term↑
  = Ann Term↓ Term↓
  | Star
  | Pi Term↓ Term↓
  | Bound Int
  | Free Name
  | Term↑ :@: Term↓
  deriving (Show, Eq)
```

We also extend the type of values with the new constructs.

```
data Value
  = VLam (Value → Value)
  | VStar
  | VPi Value (Value → Value)
  | VNeutral Neutral
```

As before, we use higher-order abstract syntax for the values, i.e., we encode binding constructs using Haskell functions. With *VPi*, we now have a new binding construct. In the dependent function space, a variable is bound that is visible in the range, but not in the domain. Therefore, the domain is simply a Value, but the range is represented as a function of type Value → Value.

Evaluation   To adapt the evaluator, we just add the two new cases for *Star* and *Pi* to the $eval_\uparrow$ function, as shown in Figure 11 (see Figure 4 for the evaluator for $\lambda_\rightarrow$). Evaluation of *Star* is trivial. For a *Pi* type, both the domain and the range are evaluated. In the range, where the bound variable x is visible, we have to add it to the environment.

Contexts   Contexts map variables to their types. Types are on the term level now. We store types in their evaluated form, and thus define:

```
type Type    = Value
type Context = [(Name, Type)]
```

$$eval_\uparrow \; Star \qquad d = VStar$$
$$eval_\uparrow \; (Pi \; \tau \; \tau') \; d = VPi \; (eval_\downarrow \; \tau \; d) \; (\lambda x \to eval_\downarrow \; \tau' \; (x : d))$$

$$subst_\uparrow \; i \; r \; (Ann \; e_\downarrow \; \tau) = Ann \; (subst_\downarrow \; i \; r \; e_\downarrow) \; (subst_\downarrow \; i \; r \; \tau)$$
$$subst_\uparrow \; i \; r \; Star \qquad = Star$$
$$subst_\uparrow \; i \; r \; (Pi \; \tau \; \tau') \quad = Pi \; (subst_\downarrow \; i \; r \; \tau) \; (subst_\downarrow \; (i+1) \; r \; \tau')$$

$$quote \; i \; VStar = Inf \; Star$$
$$quote \; i \; (VPi \; v \; f)$$
$$\quad = Inf \; (Pi \; (quote \; i \; v) \; (quote \; (i+1) \; (f \; (vfree \; (Quote \; i)))))$$

Figure 11. Extending evaluation, substitution and quotation to λ_Π

Type checking Let us go through each of the cases in Figure 12 one by one. The cases for $\lambda_\to$ – for comparison – are in Figure 5. For an annotated term, we first check that the annotation is a type of kind $*$, using the type-checking function $type_\downarrow$. We then evaluate the type. The resulting value τ is used to check the term e. If that succeeds, the entire term has type v. Note that we assume that the term under consideration in $type_\uparrow$ has no unbound variables, so all calls to $eval_\downarrow$ take an empty environment.

The (evaluated) type of $Star$ is $VStar$.

For a dependent function type, we first kind-check the domain τ. Then the domain is evaluated to v. The value is added to the context while kind-checking the range – the idea is similar to the type-checking rules for Lam in $\lambda_\to$ and λ_Π.

In the application case, the type inferred for the function is a Value now. This type must be of the form $VPi \; \tau \; \tau'$, i.e., a dependent function type. In the corresponding type rule in Figure 10, the bound variable x is substituted by e' in the result type τ'. In the implementation, τ' is a function, and the substitution is performed by applying it to the (evaluated) e'.

In the case for Inf, we have to perform the type equality check. In contrast to the type rules, we cannot compare values for equality directly in Haskell. Instead, we *quote* them and compare the resulting terms syntactically.

In the case for Lam, we require a dependent function type of form $VPi \; \tau \; \tau'$ now. As in the corresponding case for $\lambda_\to$, we add the bound variable (of type τ) to the context while checking the body. But we now perform substitution on the function body e (using $subst_\downarrow$) and on the result type τ' (by applying τ').

We thus only have to extend the substitution functions, by traversing the type in an annotated term, and by adding the two cases for $Star$ and Pi, as shown in Figure 11. There's nothing to subsitute for $Star$. For Pi, we have to increment the counter before substituting in the range because we pass a binder.

Quotation To complete our implementation of λ_Π, we only have to extend the quotation function. This operation is more important than for $\lambda_\to$, because as we have seen, it is used in the equality check during type checking. Again, we only have to add equations for $VStar$ and VPi, which are shown in Figure 11.

Quoting $VStar$ yields $Star$. Since the dependent function type is a binding construct, quotation for VPi works similar to quotation of $VLam$: to quote the range, we increment the counter i, and apply the

$type_\uparrow$ :: Int $\to$ Context $\to$ Term$_\uparrow$ $\to$ Result Type
$type_\uparrow$ i Γ $(Ann\ e\ \rho)$
    $=$ **do** $type_\downarrow$ i Γ ρ *VStar*
        let $\tau = eval_\downarrow\ \rho\ [\,]$
        $type_\downarrow$ i Γ e τ
        return τ
$type_\uparrow$ i Γ *Star*
    $=$ *return VStar*
$type_\uparrow$ i Γ $(Pi\ \rho\ \rho')$
    $=$ **do** $type_\downarrow$ i Γ ρ *VStar*
        let $\tau = eval_\downarrow\ \rho\ [\,]$
        $type_\downarrow$ $(i+1)$ $((Local\ i, \tau) : \Gamma)$
            $(subst_\downarrow\ 0\ (Free\ (Local\ i))\ \rho')$ *VStar*
        return VStar
$type_\uparrow$ i Γ $(Free\ x)$
    $=$ **case** *lookup* x Γ **of**
        Just τ $\to$ *return* τ
        Nothing $\to$ *throwError* "unknown identifier"
$type_\uparrow$ i Γ $(e :@: e')$
    $=$ **do** $\sigma \leftarrow type_\uparrow$ i Γ e
        case σ **of**
            $VPi\ \tau\ \tau' \to$ **do** $type_\downarrow$ i Γ e' τ
                            return $(\tau'\ (eval_\downarrow\ e'\ [\,]))$
            _ $\to$ *throwError* "illegal application"

$type_\downarrow$ :: Int $\to$ Context $\to$ Term$_\downarrow$ $\to$ Type $\to$ Result ()
$type_\downarrow$ i Γ $(Inf\ e)$ v
    $=$ **do** $v' \leftarrow type_\uparrow$ i Γ e
        unless $(quote_0\ v == quote_0\ v')$ $(throwError$ "type mismatch"$)$
$type_\downarrow$ i Γ $(Lam\ e)$ $(VPi\ \tau\ \tau')$
    $= type_\downarrow$ $(i+1)$ $((Local\ i, \tau) : \Gamma)$
        $(subst_\downarrow\ 0\ (Free\ (Local\ i))\ e)$ $(\tau'\ (vfree\ (Local\ i)))$
$type_\downarrow$ i Γ _ _
    $= throwError$ "type mismatch"

Figure 12.   Implementation of a type checker for λ_Π

Haskell function representing the range to *Quote i*.

3.5.   Where are the dependent types?

We now have adapted our type system and its implementation to dependent types, but unfortunately, we have not yet seen any examples.

Again, we have written a small interpreter around the type checker we have just presented, and we can use it to define and check, for instance, the polymorphic identity function (where the type argument is explicit), as follows:

$$\rangle\!\rangle \textbf{ let } id = (\lambda\alpha\ x \rightarrow x) :: \forall(\alpha :: *).\alpha \rightarrow \alpha$$
$$id :: \forall(x :: *)\ (y :: x).x$$
$$\rangle\!\rangle \textbf{ assume } (\textsf{Bool} :: *)\ (\textit{False} :: \textsf{Bool})$$

$$\rangle\!\rangle\ id\ \textsf{Bool}$$
$$\lambda x \rightarrow x :: \forall x :: \textsf{Bool}.\textsf{Bool}$$
$$\rangle\!\rangle\ id\ \textsf{Bool}\ \textit{False}$$
$$\textit{False} :: \textsf{Bool}$$

This is more than we can do in the simply-typed setting, but it is by no means spectacular and does not require dependent types. Unfortunately, while we have a framework for dependent types in place, we cannot write any interesting programs as long as we do not add at least a few specific data types to our language.

4.   Beyond λ_Π

In Haskell, data types are introduced by special data declarations:

$$\textbf{data } \textsf{Nat} = \textit{Zero} \mid \textit{Succ } \textsf{Nat}$$

This declaration introduces a new type Nat, together with two constructors *Zero* and *Succ*. In this section, we investigate how to extend our language with data types, such as natural numbers.

Obviously, we will need to add the type Nat together with its constructors; but how should we define functions, such as addition, that manipulate numbers? In Haskell, we would define a function that pattern matches on its arguments and makes recursive calls to smaller numbers:

$$plus :: \textsf{Nat} \rightarrow \textsf{Nat} \rightarrow \textsf{Nat}$$
$$plus\ Zero\ n \quad = n$$
$$plus\ (Succ\ k)\ n = Succ\ (plus\ k\ n)$$

In our calculus so far, we can neither pattern match nor make recursive calls. How could we hope to define *plus*?

In Haskell, we can define recursive functions on data types using a fold [16]. Rather than introduce pattern matching and recursion, and all the associated problems, we define functions over natural numbers using the corresponding fold. In a dependently type setting, however, we can define a slightly more general version of a fold called the *eliminator*.

$$\frac{}{\textsf{Nat} \Downarrow \textsf{Nat}} \quad \frac{}{\textit{Zero} \Downarrow \textit{Zero}} \quad \frac{k \Downarrow l}{\textit{Succ } k \Downarrow \textit{Succ } l}$$

$$\frac{k \Downarrow \textit{Zero} \quad mz \Downarrow v}{\textit{natElim } m \; mz \; ms \; k \Downarrow v} \quad \frac{k \Downarrow \textit{Succ } l \quad ms \; l \; (\textit{natElim } m \; mz \; ms \; l) \Downarrow v}{\textit{natElim } m \; mz \; ms \; k \Downarrow v}$$

$$\frac{k \Downarrow n}{\textit{natElim } m \; mz \; ms \; k \Downarrow \textit{natElim } m \; mz \; ms \; n}$$

Figure 13.   Evaluation of natural numbers

$$\frac{}{\Gamma \vdash \textsf{Nat} ::_{\uparrow} *} \quad \frac{}{\Gamma \vdash \textit{Zero} ::_{\uparrow} \textsf{Nat}} \quad \frac{\Gamma \vdash k ::_{\downarrow} \textsf{Nat}}{\Gamma \vdash \textit{Succ } k ::_{\uparrow} \textsf{Nat}}$$

$$\frac{\begin{array}{c} \Gamma \vdash m ::_{\downarrow} \textsf{Nat} \to * \\ m \; \textit{Zero} \Downarrow \tau \quad \Gamma \vdash mz ::_{\downarrow} \tau \\ \forall l :: \textsf{Nat}.m \; k \to m \; (\textit{Succ } l) \Downarrow \tau' \quad \Gamma \vdash ms ::_{\downarrow} \tau' \\ \Gamma \vdash k ::_{\downarrow} \textsf{Nat} \end{array}}{\Gamma \vdash \textit{natElim } m \; mz \; ms \; k ::_{\uparrow} m \; k}$$

Figure 14.   Typing rules for natural numbers

The fold for natural numbers has the following type:

$$\textit{foldNat} :: \forall \alpha :: *.\alpha \to (\alpha \to \alpha) \to \textsf{Nat} \to \alpha$$

This much should be familiar. In the context of dependent types, however, there is no need for the type α to be uniform across the constructors for natural numbers: rather than use $\alpha :: *$, we use $m :: \textsf{Nat} \to *$. This leads us to the following type of *natElim*:

$$\begin{aligned} \textit{natElim} :: \forall m :: \textsf{Nat} \to *. \quad & m \; \textit{Zero} \\ & \to (\forall l :: \textsf{Nat}.m \; l \to m \; (\textit{Succ } l)) \\ & \to \forall k :: \textsf{Nat}.m \; k \end{aligned}$$

The first argument of the eliminator is the sometimes referred to as the motive [10]; it explains the reason we want to eliminate natural numbers. The second argument corresponds to the base case, where k is *Zero*; the third argument corresponds to the inductive case where k is *Succ l*, for some l. In the inductive case, we must describe how to construct m (*Succ l*) from l and $m \; l$. The final argument of *natElim* is a natural number k that is eliminated, the result being of type $m \; k$.

Inspired by the discussion above, we show evaluation and typing rules for natural numbers in Figures 13 and 14. Note that we treat *natElim* not as a function, but as a term former that has to be applied to all of its arguments. The evaluation rules implement the behaviour one would expect from a fold function. We need a case to deal with the fact that the natural number k can evaluate to a neutral term. In this case, evaluation of the eliminator is stuck and we return a neutral term containing the application of the eliminator.

$eval_\uparrow$ *Nat* $d = VNat$
$eval_\uparrow$ *Zero* $d = VZero$
$eval_\uparrow$ (*Succ k*) $d = VSucc$ ($eval_\downarrow$ *k d*)
$eval_\uparrow$ (*NatElim m mz ms k*) *d*
  = **let** *mzVal* = $eval_\downarrow$ *mz d*
      msVal = $eval_\downarrow$ *ms d*
      rec kVal =
        case *kVal* **of**
          VZero $\rightarrow$ *mzVal*
          VSucc l $\rightarrow$ *msVal* 'vapp' *l* 'vapp' *rec l*
          VNeutral k $\rightarrow$ *VNeutral*
                    (*NNatElim* ($eval_\downarrow$ *m d*) *mzVal msVal k*)
          _ $\rightarrow$ *error* `"internal: eval natElim"`
    in *rec* ($eval_\downarrow$ *k d*)

Figure 15.   Extending the evaluator for natural numbers

4.1.   Implementing natural numbers

In summary, adding natural numbers to our language involves adding three separate elements: the type Nat, the constructors *Zero* and *Succ*, and the eliminator *natElim*. To implement these three components, We extend the abstract syntax and correspondingly add new cases to the evaluation and type checking functions. These new cases do not require any changes to existing code; we choose to focus only on the new code fragments.

Abstract Syntax   To implement natural numbers, we extend our abstract syntax as follows:

    data Term$_\uparrow$ = . . .
      | *Nat*
      | *NatElim* Term$_\downarrow$ Term$_\downarrow$ Term$_\downarrow$ Term$_\downarrow$
      | *Zero*
      | *Succ* Term$_\downarrow$

We add four constructors to Term$_\uparrow$: *Nat* for the datatype, *Zero* and *Succ* for the data constructors, and *NatElim* for the eliminator. The *NatElim* constructor is fully applied: it expects no further arguments.

Evaluation   We need to rethink our data type for values. Previously, values consisted exclusively of lambda abstractions and 'stuck' applications. Clearly, we will need to extend the data type for values to cope with the new constructors for natural numbers.

    data Value = . . .
      | *VNat*
      | *VZero*
      | *VSucc* Value

```
type↑ i Γ Nat              = return VStar
type↑ i Γ Zero             = return VNat
type↑ i Γ (Succ k)         =
   do type↓ i Γ k VNat
      return VNat
type↑ i Γ (NatElim m mz ms k) =
   do type↓ i Γ m (VPi VNat (const VStar))
      let mVal = eval↓ m [ ]
      type↓ i Γ mz (mVal 'vapp' VZero)
      type↓ i Γ ms (VPi VNat (λl → VPi (mVal 'vapp' l) (λ_ → mVal 'vapp' VSucc l)))
      type↓ i Γ k VNat
      let kVal = eval↓ k [ ]
      return (mVal 'vapp' kVal)
```

Figure 16.   Extending the type checker for natural numbers

Introducing the eliminator, however, also complicates evaluation. The eliminator for natural numbers can also be stuck when the number being eliminated does not evaluate to a constructor. Correspondingly, we extend the data type for neutral terms to cover this case:

```
data Neutral = ...
   | NNatElim Value Value Value Neutral
```

The implementation of evaluation in Figure 15 closely follows the rules in Figure 13. The eliminator is the only interesting case. Essentially, the eliminator evaluates to the Haskell function with the behaviour you would expect: if the number being eliminated evaluates to *VZero*, we evaluate the base case *mz*; if the number evaluates to *VSucc l*, we apply the step function *ms* to the predecessor *l* and the recursive call to the eliminator; finally, if the number evaluates to a neutral term, the entire expression evaluates to a neutral term. If the value being eliminated is not a natural number or a neutral term, this would have already resulted in a type error. Therefore, the final catch-all case should never be executed.

Typing   Figure 16 contains the implementation of the type checker that deals with natural numbers. Checking that *Zero* and *Succ* construct natural numbers is straightforward.

Type checking the eliminator is bit more involved. Remember that the eliminator has the following type:

$$natElim :: \forall m :: \mathsf{Nat} \to *. \quad m\ Zero$$
$$\to (\forall l :: \mathsf{Nat}.m\ l \to m\ (Succ\ l))$$
$$\to \forall k :: \mathsf{Nat}.m\ k$$

We begin by type checking and evaluating the motive *m*. Once we have the value of *m*, we type check the two branches. The branch for zero should have type *m Zero*; the branch for successors should have type $\forall l :: \mathsf{Nat}.m\ l \to m\ (Succ\ l)$. Despite the apparent complication resulting from having to hand code

complex types, type checking these branches is exactly what would happen when type checking a fold over natural numbers in Haskell. Finally, we check that the *k* we are eliminating is actually a natural number. The return type of the entire expression is the motive, accordingly applied to the number being eliminated.

Other functions   To complete the implementation of natural numbers, we must also extend the auxiliary functions for substitution and quotations with new cases. All new code is, however, completely straightforward, because no new binding constructs are involved.

Addition   With all the ingredients in place, we can finally define addition in our interpreter as follows:

$\rangle\!\rangle$ **let** *plus* = *natElim* ($\lambda_- \rightarrow$ Nat $\rightarrow$ Nat)
$\qquad\qquad\qquad\quad$ ($\lambda n \rightarrow n$)
$\qquad\qquad\qquad\quad$ ($\lambda k\ rec\ n \rightarrow Succ\ (rec\ n)$)
$\quad$ *plus* :: $\forall(x ::$ Nat$)\ (y ::$ Nat$)$.Nat

We define a function *plus* by eliminating the first argument of the addition. In each case branch, we must define a function of type Nat $\rightarrow$ Nat; we choose our motive correspondingly. In the base case, we must add zero to the argument *n* – we simply return *n*. In the inductive case, we are passed the predecessor *k*, the recursive call *rec* (that corresponds to adding *k*), and the number *n*, to which we must add *Succ k*. We proceed by adding *k* to *n* using *rec*, and wrapping an additional *Succ* around the result. After having defined *plus*, we can evaluate simple additions in our interpreter:[5]

$\rangle\!\rangle$ *plus* 40 2
42 :: Nat

4.2.   Implementing vectors

Natural numbers are still not particularly exciting: they are still the kind of data type we can write quite easily in Haskell. As an example of a data type that really makes use of dependent types, we show how to implement vectors.

As was the case for natural numbers, we need to define three separate components: the type of vectors, its constructors, and the eliminator. We have already mentioned that vectors are parameterized by both a type and a natural number:

$\forall \alpha ::\ *.\forall k ::$ Nat.Vec $\alpha\ k ::\ *$

The constructors for vectors are analogous to those for Haskell lists. The only difference is that their types record the length of the vector:

Nil$\quad$ :: $\forall \alpha ::\ *$.Vec α *Zero*
Cons :: $\forall \alpha ::\ *.\forall k ::$ Nat.$\alpha \rightarrow$ Vec $\alpha\ k \rightarrow$ Vec α (*Succ k*)

---

[5]For convenience, our parser and pretty-printer support literals for natural numbers. For instance, 2 is translated to *Succ* (*Succ Zero*) :: Nat on the fly.

The eliminator for vectors behaves essentially the same as *foldr* on lists, but its type is a great deal more specific (and thus, more involved):

$$vecElim :: \forall \alpha :: *.\forall m :: (\forall k :: \mathsf{Nat}.\mathsf{Vec}\ \alpha\ k \rightarrow *).$$
$$m\ Zero\ (Nil\ \alpha)$$
$$\rightarrow (\forall l :: \mathsf{Nat}.\forall x :: \alpha.\forall xs :: \mathsf{Vec}\ \alpha\ l.$$
$$m\ l\ xs \rightarrow m\ (Succ\ l)\ (Cons\ \alpha\ l\ x\ xs))$$
$$\rightarrow \forall k :: \mathsf{Nat}.\forall xs :: \mathsf{Vec}\ \alpha\ k.m\ k\ xs$$

The whole eliminator is quantified over the element type α of the vectors. The next argument of the eliminator is the motive. As was the case for natural numbers, the motive is a type (kind $*$) parameterized by a vector. As vectors are themselves parameterized by their length, the motive expects an additional argument of type Nat. The following two arguments are the cases for the two constructors of Vec. The constructor *Nil* is for empty vectors, so the corresponding argument is of type $m\ Zero\ (Nil\ \alpha)$. The case for *Cons* takes a number l, an element x of type α, a vector xs of length l, and the result of the recursive application of the eliminator of type $m\ l\ xs$. It combines those elements to form the required type, for the vector of length *Succ l* where x has been added to xs. The final result is a function that eliminates a vector of any length.

The type of the eliminator may look rather complicated. However, if we compare with the type of *foldr* on lists

$$foldr :: \forall \alpha :: *.\forall m :: *.m \rightarrow (\alpha \rightarrow m \rightarrow m) \rightarrow [\alpha] \rightarrow m$$

we see that the structure is the same, and the additional complexity stems only from the fact that the motive is parameterized by a vector, and vectors are in turn parameterized by natural numbers.

Not all of the arguments of *vecElim* are actually required – some of the arguments can be inferred from others, to reduce the noise and make writing programs more feasible. We would like to remind you that λ_Π is designed to be a very explicit, low-level language.

Abstract syntax   As was the case for natural numbers, we extend the abstract syntax. We add the type of vectors, the vector constructors, and the eliminator to $\mathsf{Term}_\uparrow$.

```
data Term↑ = ...
    | Vec Term↓ Term↓
    | Nil Term↓
    | Cons Term↓ Term↓ Term↓ Term↓
    | VecElim Term↓ Term↓ Term↓ Term↓ Term↓ Term↓
```

Note that also *Nil* takes an argument, because both constructors are polymorphic in the element type. For vectors and for many other datatypes there is some flexibility as to where the constructors are placed: we could omit the type argument from *Nil* and *Cons* and the length argument from *Cons* and place the constructors in $\mathsf{Term}_\downarrow$ rather than $\mathsf{Term}_\uparrow$. We would have to place less arguments on the individual constructor applications at the price that we have to explicitly annotate vector expressions.

We also extend the data types for values and neutral terms:

eval↑ (*VecElim α m mn mc k xs*) *d* =
  let *mnVal* = *eval*↓ *mn d*
    mcVal = *eval*↓ *mc d*
    rec kVal xsVal =
      case *xsVal* **of**
        VNil _   → *mnVal*
        VCons _ *l x xs* → *foldl vapp mcVal* [*l, x, xs, rec l xs*]
        VNeutral n   → *VNeutral*
                    (*NVecElim* (*eval*↓ *α d*) (*eval*↓ *m d*)
                          mnVal mcVal kVal n)
        _   → *error* `"internal: eval vecElim"`
  in *rec* (*eval*↓ *k d*) (*eval*↓ *xs d*)

Figure 17.   Implementation of the evaluation of vectors

data Value = . . .
    | *VNil* Value
    | *VCons* Value Value Value Value
    | *VVec* Value Value

data Neutral = . . .
    | *NVecElim* Value Value Value Value Value Neutral

Evaluation   Evaluation of constructors of the Vec type proceeds structurally, turning terms into their value counterparts. Once again, the only interesting case is the evaluation of the eliminator for vectors, shown in Figure 17. As indicated before, the behaviour resembles a fold on lists: depending on whether the vector is a *VNil* or a *VCons*, we apply the appropriate argument. In the case for *VCons*, we also call the eliminator recursively on the tail of the vector (of length *l*). If the eliminated vector is a neutral element, we cannot reduce the eliminator, and produce a neutral term again.

Type checking   We extend the type checker as shown in Figure 18. The code is relatively long, but keeping the types of each of the constructs in mind, there are absolutely no surprises.

    As for natural numbers, we have omitted the new cases for substitution and quotation, because they are entirely straightforward.

Append   We are now capable of demonstrating a real dependently typed program in action, a function that appends two vectors while keeping track of their lengths. The definition in the interpreter looks as follows:

type↑ *i* Γ (*Vec α k*) =
  do *type*↓ *i* Γ *α VStar*
     type↓ *i* Γ *k VNat*
     return VStar
type↑ *i* Γ (*Nil α*) =
  do *type*↓ *i* Γ *α VStar*
     let *aVal* = *eval*↓ *α* [ ]
     return (*VVec aVal VZero*)
type↑ *i* Γ (*Cons α k x xs*) =
  do *type*↓ *i* Γ *α VStar*
     let *aVal* = *eval*↓ *α* [ ]
     type↓ *i* Γ *k VNat*
     let *kVal* = *eval*↓ *k* [ ]
     type↓ *i* Γ *x aVal*
     type↓ *i* Γ *xs* (*VVec aVal kVal*)
     return (*VVec aVal* (*VSucc kVal*))
type↑ *i* Γ (*VecElim α m mn mc k vs*) =
  do *type*↓ *i* Γ *α VStar*
     let *aVal* = *eval*↓ *α* [ ]
     type↓ *i* Γ *m*
        (*VPi VNat* (*λk* → *VPi* (*VVec aVal k*) (*λ*_ → *VStar*)))
     let *mVal* = *eval*↓ *m* [ ]
     type↓ *i* Γ *mn* (*foldl vapp mVal* [*VZero, VNil aVal*])
     type↓ *i* Γ *mc*
        (*VPi VNat* (*λl* →
        VPi aVal (*λy* →
        VPi (*VVec aVal l*) (*λys* →
        VPi (*foldl vapp mVal* [*l, ys*]) (*λ*_ →
        (*foldl vapp mVal* [*VSucc l, VCons aVal l y ys*]))))))
     type↓ *i* Γ *k VNat*
     let *kVal* = *eval*↓ *k* [ ]
     type↓ *i* Γ *vs* (*VVec aVal kVal*)
     let *vsVal* = *eval*↓ *vs* [ ]
     return (*foldl vapp mVal* [*kVal, vsVal*])

Figure 18.   Extending the type checker for vectors

⟫ **let** *append* =
      (λα → *vecElim* α
                  (λ*m* _ → ∀(*n* :: Nat).Vec α *n* → Vec α (*plus m n*))
                  (λ_ *v* → *v*)
                  (λ*m v vs rec n w* → *Cons* α (*plus m n*) *v* (*rec n w*)))
      :: ∀(α :: ∗) (*m* :: Nat) (*v* :: Vec α *m*) (*n* :: Nat) (*w* :: Vec α *n*).
          Vec α (*plus m n*)

Like for *plus*, we define a binary function on vectors by eliminating the first argument. The motive is chosen to expect a second vector. The length of the resulting vector is the sum of the lengths of the argument vectors *plus m n*. Appending an empty vector to another vector *v* results in *v*. Appending a vector of the form *Cons m v vs* to a vector *v* works by invoking recursion via *rec* (which appends *vs* to *w*) and prepending *v*. Of course, we can also apply the function thus defined:

⟫ **assume** (α :: ∗) (*x* :: α) (*y* :: α)

⟫ *append* α 2 (*Cons* α 1 *x* (*Cons* α 0 *x* (*Nil* α)))
              1 (*Cons* α 0 *y* (*Nil* α))
Cons α 2 *x* (*Cons* α 1 *x* (*Cons* α 0 *y* (*Nil* α))) :: Vec α 3

We assume a type α with two elements *x* and *y*, and append a vector containing two *x*'s to a vector containing one *y*.

4.3. Discussion

In this section we have shown how to add two data types to our core theory: natural numbers and vectors. Using exactly the same principles, many more data types can be added. For example, for any natural number *n*, we can define the type Fin *n* that contains exactly *n* elements. In particular, Fin 0, Fin 1 and Fin 2 are the empty type, the unit type, and the type of booleans respectively. Furthermore, Fin can be used to define a total projection function from vectors, of type

project :: ∀(α :: ∗) (*n* :: Nat).Vec α *n* → Fin *n* → α

Another interesting dependent type is the equality type

Eq :: ∀(α :: ∗).α → α → ∗

with a single constructor

Refl :: ∀(α :: ∗) (*x* :: α) → Eq α *x x*

and eliminator

eqElim :: ∀(α :: ∗).∀(*m* :: ∀(*x* :: α).∀(*y* :: α).Eq α *x y* → ∗).
                  (∀(*z* :: α).*m z z* (*Refl* α *z*))
              → ∀(*x* :: α).∀(*y* :: α).∀(*p* :: Eq α *x y*).*m x y p*

Using Eq, we can state and prove theorems about our code directly in λ_Π. For instance, the type

$$\forall (\alpha :: *) \ (n :: \mathsf{Nat}).\mathsf{Eq} \ \mathsf{Nat} \ (plus \ n \ Zero) \ n$$

states that *Zero* is the right-neutral element of addition. Any term of that type serves as a proof of that theorem, via the Curry-Howard isomorphism.

These examples and a few more are included with the interpreter in the paper sources, which can be downloaded via the λ_Π homepage [7]. More about suitable data types for dependently typed languages and writing dependently typed programs can be found in another tutorial [12].

Throughout this section, we have chosen to extend the abstract syntax of our language for every data type we add. Alternatively, we could use the *Church encoding* of data types, e.g., representing natural numbers by the type $\forall (\alpha :: *).\alpha \to (\alpha \to \alpha) \to \alpha$. Although this choice may seem to require less effort, it does introduce some problems. Although we can use the Church encoding to write simple folds, we cannot write dependently typed programs that rely on eliminators without extending our theory further. This makes it harder to write programs with an inherently dependent type, such as our *append* function. As our core theory should be able to form the basis of a dependently typed programming language, we chose to avoid using such an encoding.

5.   Toward dependently typed programming

The calculus we have described is far from a real programming language. Although we can write, type check, and evaluate simple expressions there is still a lot of work to be done before it becomes feasible to write large, complex programs. In this section, we do not strive to enumerate all the problems that large-scale programming with dependent types must face, let alone solve them. Instead, we try to sketch how a programming language may be built on top of the core calculus we have seen so far and point you to related literature.

As our examples illustrate, programming with eliminators does not scale. Epigram [15] uses a clever choice of motive to make programming with eliminators a great deal more practical [9, 14]. By choosing the right motive, we can exploit type information when defining complicated functions. Eliminators may not appear to be terribly useful, but they form the foundations on which dependently typed programming languages may be built.

Writing programs with complex types in one go is not easy. Epigram and Agda [19] allow programmers to put 'holes' in their code, leaving parts of their programs undefined [20]. Programmers can then ask the system what type a specific hole has, effectively allowing the incremental development of complex programs.

As it stands, the core system we have presented requires programmers to explicitly instantiate polymorphic functions. This is terribly tedious! Take the *append* function we defined: of its five arguments, only two are interesting. Fortunately, uninteresting arguments can usually be inferred. Many programming languages and proof assistants based on dependent types have support for *implicit arguments* that the user can omit when calling a function. Note that these arguments need not be types: the *append* function is 'polymorphic' in the length of the vectors.

Finally, we should reiterate that the type system we have presented is unsound. As the kind of $*$ is itself $*$, we can encode a variation of Russell's paradox, known as Girard's paradox [3]. This allows us to create an inhabitant of any type. To fix this, the standard solution is to introduce an infinite hierarchy of types: the type of $*$ is $*_1$, the type of $*_1$ is $*_2$, and so forth.

6.   Discussion

There is a large amount of relevant literature regarding both implementing type systems and type theory. Pierce's book [21] is an excellent place to start. Martin-Löf's notes on type theory [8] are still highly relevant and form an excellent introduction to the subject. More recent books by Nordström et al. [17] and Thompson [24] are freely available online.

There are several dependently typed programming languages and proof assistants readily available. Coq [2] is a mature, well-documented proof assistant. While it is not primarily designed for dependently typed *programming*, learning Coq can help get a feel for type theory. Haskell programmers may feel more at home using recent versions of Agda [18], a dependently typed programming language. Not only does the syntax resemble Haskell, but functions may be defined using pattern matching and general recursion. Finally, Epigram [15, 12] proposes a more radical break from functional programming as we know it. While the initial implementation is far from perfect, many of Epigram's ideas are not yet implemented elsewhere.

Other implementations of the type system we have presented here have been published elsewhere [1, 4, 5]. These articles, however, tend to focus on proving that an implementation is correct and satisfies certain meta-theoretical properties. The focus of our paper is somewhat different: we have chosen to describe a concrete implementation of a type checker as a vehicle for explanation. Furthermore, we have tried to collect and document those implementation techniques that we found useful in our implementation.

In the introduction we mentioned some of the concerns functional programmers have regarding dependent types. Type checking a language with dependent types is not necessarily undecidable – indeed, the type checker we have presented here will only fail to terminate for certain contrived examples [3]. The phase distinction between evaluation and type checking becomes more subtle, but is not lost. The fusion of types and terms introduces new challenges, but also has a lot to offer. Most importantly, though, getting started with dependent types is not as hard as you may think. We hope to have whet your appetite, guiding you through your first steps, but encourage you to start exploring dependent types yourself!

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